



## Local approaches for the fracture assessment of notched components: the research work developed by Professor Paolo Lazzarin

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**ABSTRACT.** Brittle failure of components weakened by cracks or sharp and blunt V-notches is a topic of active and continuous research. It is attractive for all researchers who face the problem of fracture of materials under different loading conditions and deals with a large number of applications in different engineering fields, not only with the mechanical one. This topic is significant in all the cases where intrinsic defects of the material or geometrical discontinuities give rise to localized stress concentration which, in brittle materials, may generate a crack leading to catastrophic failure or to a shortening of the assessed structural life. Whereas cracks are viewed as unpleasant entities in most engineering materials, U- and V-notches of different acuties are sometimes deliberately introduced in design and manufacturing of structural components.

Dealing with brittle failure of notched components and summarising some recent experimental results reported in the literature, the main aim of the present contribution is to present a review of the research work developed by Professor Paolo Lazzarin. The approach based on the volume strain energy density (SED), which has been recently applied to assess the brittle failure of a large number of materials. The main features of the SED approach are outlined in the paper and its peculiarities and advantages accurately underlined. Some examples of applications are reported, as well. The present contribution is based on the author's experience over about 15 years and the contents of his personal library. This work is in honor and memory of Prof. Paolo Lazzarin who suddenly passed away in September 2014.

**KEYWORDS.** Notch stress intensity factors; Strain Energy Density; Fracture assessment; Fatigue strength.

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### INTRODUCTION

Dealing with fracture assessment of cracked and notched components a clear distinction should be done between large and small bodies [1-6]. The design rules applied to large bodies are based on the idea that local inhomogeneities, where material damage starts, can be averaged being large the volume to surface ratio. In small bodies the high ratio between surface and volume makes not negligible the local discontinuities present in the material and the adoption of a multi-scaling and segmentation scheme is the only way to capture what happens at pico, nano and micro levels [4-6]. In this scheme the crack tip has no dimension or mass to speak; it is the sink and source that absorbs and dissipates energy while the stress singularity representation at every level is the most powerful tool to quantify the energy packed by an equivalent crack reflecting both material effect and boundary conditions. This new revolutionary

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representation implies also a new definition of mass [4, 6]. The distinction between large and small bodies should ever be considered by avoiding to transfer directly the design rules valid for large components to small ones under the hypothesis that all material inhomogeneities can be averaged [1-6].

Keeping in mind the observations above and limiting our considerations to large bodies (i.e. large volume to surface ratio), for which an averaging process is still valid, the paper is addressed to review a volume-based Strain Energy Density Approach applied to Static and Fatigue Strength Assessments of notched and welded structures [7-14].

The concept of “elementary” volume and “structural support length” was introduced many years ago [15-17] and it states that not the theoretical maximum notch stress is the static or fatigue strength-effective parameter in the case of pointed or sharp notches, but rather the notch stress averaged over a short distance normal to the notch edge. In high cycle fatigue regime, the integration path should coincide with the early fatigue crack propagation path. A further idea was to determine the fatigue-effective notch stress directly (i.e. without notch stress averaging) by performing the notch stress analysis with a fictitiously enlarged notch radius,  $\rho_f$ , corresponding to the relevant support [15-17].

Fundamentals of Critical Distance Mechanics applied to static failure, state that crack propagation occurs when the normal strain [18] or circumferential stress  $\sigma_{\theta\theta}$  [19] at some critical distance from the crack tip reaches a given critical value. This “Point Criterion” becomes a “Line criterion” in Refs. [20, 21] who dealt with components weakened by sharp V-shaped notches. A stress criterion of brittle failure was proposed based on the assumption that crack initiation or propagation occurs when the mean value of decohesive stress over a specified damage segment  $d_0$  reaches a critical value. The length  $d_0$  is 2-5 times the grain size and then ranges for most metals from 0.03 mm to 0.50 mm. The segment  $d_0$  was called “elementary increment of the crack length”. Dealing with this topic a previous paper, Ref. [22], was quoted in Refs [20, 21]. Afterwards, this critical distance-based criterion was extended also to structural elements under multi-axial loading [23, 24] by introducing a non-local failure function combining normal and shear stress components, both normalised with respect to the relevant fracture stresses of the material.

Dealing with notched components the idea that a quantity averaged over a finite size volume controls the stress state in the volume by means of a single parameter, the average value of the circumferential  $\sigma_{\theta\theta}$  stress, [25].

For many years the Strain Energy Density (SED) has been used to formulate failure criteria for materials exhibiting both ductile and brittle behaviour. Since Beltrami [26] to nowadays the SED has been found being a powerful tool to assess the static and fatigue behaviour of notched and un-notched components in structural engineering. Different SED-based approaches were formulated by many researchers.

Dealing here with the strain energy density concept, it is worthwhile contemplating some fundamental contributions [27-35]. The concept of “core region” surrounding the crack tip was proposed in Ref. [27]. The main idea is that the continuum mechanics stops short at a distance from the crack tip, providing the concept of the radius of the core region. The strain energy density factor  $S$  was defined as the product of the strain energy density by a critical distance from the point of singularity [28]. Failure was thought of as controlled by a critical value  $S_c$ , whereas the direction of crack propagation was determined by imposing a minimum condition on  $S$ . The theory was extended to employ the total strain energy density near the notch tip [29], and the point of reference was chosen to be the location on the surface of the notch where the maximum tangential stress occurs. The strain energy density fracture criterion was refined and extensively summarised in Ref. [30]. The material element is always kept at a finite distance from the crack or the notch tip outside the “core region” where the in-homogeneity of the material due to micro-cracks, dislocations and grain boundaries precludes an accurate analytical solution. The theory can account for yielding and fracture and is applicable also to ductile materials. Depending on the local stress state, the radius of the core region may or may not coincide with the critical ligament  $r_c$  that corresponds to the onset of unstable crack extension [30]. The ligament  $r_c$  depends on the fracture toughness  $K_{IC}$ , the yield stress  $\sigma_y$ , the Poisson’s ratio  $\nu$  and, finally, on the ratio between dilatational and distortional components of the strain energy density. The direction of  $\sigma_{\max}$  determines maximum distortion while  $\sigma_{\min}$  relates to dilatation. Distortion is associated with yielding, dilatation tends to be associated to the creation of free surfaces or fracture and occurs along the line of expected crack extension [30, 31].

A critical value of strain energy density function  $(dW/dV)_c$  has been extensively used since 1965 [32-35], when first the ratio  $(dW/dV)_c$  was determined experimentally for various engineering materials by using plain and notched specimens. The deformation energy required for crack initiation in a unit volume of material is called Absorbed Specific Fracture Energy (ASF<sub>E</sub>) and its links with the critical value of  $J_c$  and the critical factor  $S_c$  were widely discussed. This topic was deeply considered in Refs. [28-30] where it was showed that  $(dW/dV)_c$  is equivalent to  $S_c/r$  being  $S_c$  the critical strain energy density factor and the radius vector  $r$  the location of failure. Since distributions of the absorbed specific energy  $W$  in notched specimens are not uniform, it was assumed that the specimen cracks as soon as a precise energy amount has been absorbed by the small plastic zone at the root of the notch. If the notch is sufficiently sharp, specific energy due to the elastic deformation is small enough to be neglected as an initial approximation [34]. While measurements of the energy



in an infinitely small element are not possible, they can be approximated with sufficient accuracy by calculating the fracture energy over the entire fractured cross section of an unnotched tensile specimen [35]. Notched components loaded under static loads show that the average ASFE decreases with increasing the notch sharpness, with the ASFE parameter being plotted as a function of the theoretical stress concentration factor,  $K_t$ , and the temperature [35]. For a common welded structural steel and  $K_t = 1$ , the ASFE value, obtained by tensile tests, is about  $1.0 \text{ MJ/m}^3$  while for values of  $K_t$  greater than 3.0 a plateau value is visible [35]. Depending on the considered welded metal, the plateau approximately ranges between 0.15 and  $0.35 \text{ MJ/m}^3$ . These values are not so different from the mean value that characterises the high cycle fatigue strength of welded joints,  $W_c = 0.105 \text{ MJ/m}^3$  but with reference to a specific control volume [8, 10].

The criterion based on the energy density factor,  $S$ , gave a sound theoretical basis to the experimental findings [32-35] and the approach, used in different fields, was strongly supported by a number of researchers [36].

The concept of strain energy density has also been reported in the literature in order to predict the fatigue behaviour of notches both under uniaxial and multi-axial stresses [37-38].

It should be remembered that in referring to small-scale yielding, a method based on the averaged of the stress and strain product within the elastic-plastic domain around the notch was extended to cyclic loading of notched components [39]. In particular in Ref. [40] it was proposed a *fatigue master life curve* based on the use of the plastic strain energy per cycle as evaluated from the cyclic *hysteresis loop* and the positive part of the elastic strain energy density. The two views, cyclic hysteresis loop concept evaluating the plastic energy for tensile specimens [39, 40] and the criterion evaluating the local accumulated SED near the crack tip [28], although formally different, are strictly connected and both tied to the concept of Absorbed Specific Fracture Energy.

The averaged strain energy density criterion, proposed in Refs [7-14, 41], states that brittle failure occurs when the mean value of the strain energy density over a control volume (which becomes an area in two dimensional cases) is equal to a critical energy  $W_c$ . The SED approach is based both on a precise definition of the control volume and the fact that the critical energy does not depend on the notch sharpness. Such a method was formalised and applied first to sharp, zero radius, V-notches and later extended to blunt U- and V-notches under Mode I loading [11] and successfully applied to welded joints [10]. The control radius  $R_0$  of the volume, over which the energy has to be averaged, depends on the ultimate tensile strength, the fracture toughness and Poisson's ratio in the case of static loads, whereas it depends on the unnotched specimen's fatigue limit, the threshold stress intensity factor range and the Poisson's ratio under high cycle fatigue loads. The approach was successfully used under both static and fatigue loading conditions to assess the strength of notched and welded structures subjected to predominant mode I and also to mixed mode loading [7-14]. The extension of the SED approach to ductile fracture is possible, with a major problem being the definition of the control volume and the influence of the dilatational and distortional components of the strain energy density. Recently, the effect of plasticity in terms of strain energy density over a given control volume has been considered by the present authors, showing different behaviours under tension and torsion loading, as well as under small and large scale yielding [14].

Several criteria have been proposed to predict fracture loads of components with *notches*, subjected to *mode I loading* [20-21, 42-53]. Recently, fracture loads of notched specimens (sharp and blunted U and V notches) loaded under mode I have been successfully predicted, using a criterion based on the cohesive zone model [54-57], and in parallel by applying the local strain energy density [7-14]. The problem of brittle failure from *blunted notches* loaded under *mixed mode* is more complex than in mode I loading and experimental data, particularly for notches with a non-negligible radius, is scarce. The main aim of some recent papers was to generalise the previous results valid for components with blunted notches loaded under mode I, to notched components loaded under mixed mode [58-61]. This generalization is based on the hypothesis that fracture mainly depends on the local mode I and on the maximum value of the principal stress or the strain energy density. The proposal of mode I dominance for cracked plates was suggested first in Ref. [62] when dealing with cracked plates under plane loading and transverse shear, where the crack grows in the direction almost perpendicular to the maximum tangential stress in radial direction from its tip. Two different methods are used to verify such a hypothesis: the cohesive zone model and the model based on the strain energy density over a control volume [58-60]. Both methods allow us to evaluate the critical load under different mixed mode conditions when the material behaviour can be assumed as linear elastic. Dealing with the SED approach it is worth noting that the case of pure compression or combined compression and shear, for example, would require a reformulation for the control radius of the volume,  $R_0$ , and should also take into account the variability of the critical strain energy density  $W_c$  with respect to the case of uniaxial tension loads. To the best of Author's knowledge, the first contribution that modifies the total strain energy density criterion (Beltrami's hypothesis) to account for the different strength properties exhibited by many materials under pure tension and pure compression uniaxial tests was dated 1926 [63].

Dealing with both notched and welded components and summarising the most recent experimental results reported in the literature, the main aim of the present contribution is to present a complete review of the analytical frame of the volume-



based SED approach together with a final synthesis of more than 1900 experimental data from static and fatigue tests. Very different materials have been considered with a control radius,  $R_0$ , ranging from 0.4  $\mu\text{m}$  to 500  $\mu\text{m}$ . A wider and complete synthesis of this work can be found in [64].

## A SUMMARY OF THE CAREER OF PROF. PAOLO LAZZARIN

**P**aolo Lazzarin, Professor at the University of Padova, Vicenza (Italy), outstanding scientist in theoretical and applied notch mechanics, has passed away on September 14, 2014.

Paolo Lazzarin was born on April 21, 1957 in Conegliano near Treviso (Italy). He was graduated at the University of Padova, Faculty of Mechanical Engineering, in 1982 and obtained his PhD in 'Mechanical Behaviour of Materials' at the University of Pisa in 1986. In 1992 he became associate professor of 'Machine Design' at the University of Cassino and, afterwards, at the University of Ferrara. Since 2000 he has been full professor of 'Machine Design' at the University of Padova, Department of Management and Engineering in Vicenza. In this position, he has presided the master course in Mechanical Engineering since 2012.

Paolo Lazzarin is the author of about 100 significant scientific articles in renown scientific journals. Additionally, he has gained merit in the publication sector as a reviewer on demand of these journals. Also, he was a member of the board of the journal 'Fatigue and Fracture of Engineering Materials and Structures' since 2002 and became its co-editor in 2012.

Paolo Lazzarin was also the organiser of international conferences which aroused considerable attention – the Crack Path Conference in 2009 and the Mesomechanics Conference in 2011, both conferences held in Vicenza.

The appreciation hereafter of Paolo Lazzarin's scientific achievements cannot specify the respective collaborators and coauthors because of space restrictions. More than 200 pioneering articles in scientific journals are available, all written together with well-known colleagues or talented disciples. Paolo Lazzarin's scientific achievements can be assigned to three areas of notch mechanics: notch stress intensity factors, local strain energy density and fictitious notch rounding.

Paolo's scientific career started in the mid-eighties of the last century with Bruno Atzori as his mentor. His challenge was the mathematical analysis of the stress field at fillet weld toes, modelling them as pointed V-notches for assessment of the fatigue limit of these joints within a worst case scenario. The asymptotic stress distribution ahead of the stress singularity at pointed V-notches was available from Williams' linear-elastic solution. It was later on quantified by means of notch stress intensity factors (NSIFs), separating the contribution of the three loading modes (in-plane tensile and shear loading as well as out-of-plane shear loading) to the overall stress distribution ahead of the V-notch tip. The NSIFs were derived for weld-like geometries and cross-sectional models of various fillet-welded joints. The scale effect, i.e. the influence of specimen size on the level of the asymptotic stresses is already included in the NSIFs.

Fatigue strength values in terms of the NSIF amplitudes dependent on number of loading cycles for fillet-welded joints made of steel or aluminium alloys were presented as a scatterband, evaluating a large body of test results in the open literature. The scatterband widths and the curve gradients were well in agreement with comparable nominal stress relationships in the codes. Additionally, a mixed-mode failure criterion in terms of the NSIFs was derived. Also, the relationship to the standardised structural stress at the weld toe was clarified.

The elementary NSIF concept for pointed notches has been extended by Paolo Lazzarin to sharply rounded (blunt) or root-holed V-notches. Notch rounding removes the stress singularity, but the asymptotic stress distribution connected with the singularity remains largely unchanged at distances from the notch root larger than one half of the notch radius. It was shown that generalised NSIFs can be defined as the governing field parameters. These are related to the maximum notch stresses which constitute the conventional stress concentration factors (SCFs). In contrast to the SCFs, the generalised NSIFs characterise not only the maximum stress at the notch root but the whole stress field in the vicinity of the notch root. The field information is needed for assessing local failure processes, but it is not possible in general to express the fatigue limit of sharply rounded notches solely by critical NSIF amplitudes. A remarkable by-product of these efforts were the in-plane and out-of-plane stress field solutions for V-notches with root hole.

Under certain conditions such as low sheet thickness in lap joints, the radial distance from the slit tip, where the second order approximation of the stresses (inclusive of the T-stress) is appropriate, may be very small, so that higher order approximations are needed. Stress equations for the in-plane loading modes at slit tips up to the seventh order have been derived.

The conventional loading modes 1, 2 and 3 refer to singular stress fields at V-notches which can be considered as two-dimensional (plane or anti-plane). It is presumed that the corresponding NSIFs do not vary along the straight notch tip line. At the intersection with lateral free surfaces, an abrupt change of the homogeneous conditions takes place. A complex three-dimensional, generally singular stress state occurs near the intersection point. It may be described by the





applied primary loading mode, locally changed and occasionally superimposed by an induced secondary loading mode. These local coupled effects were analysed by the FE method, using extremely fine meshes near the intersection point. Paolo Lazzarin has also presented an analytical solution for the V-notch in plates of finite thickness under plane-strain conditions.

The linear-elastic NSIF concept has been extended by Paolo Lazzarin to elastic-plastic material behaviour. The stress singularity at pointed notches continues to exist, provided strain hardening occurs. It is described by plastic NSIFs with inclusion of plastic strain intensity factors which exist also for non-hardening material behaviour. The theoretical basis is the nonlinear power-law applied to crack tips by Hutchinson, Rice and Rosengreen. The HRR theory was reformulated by Paolo Lazzarin for application to V-notches. Neuber's nonlinear material law, Glinka's alternative procedure as well as the Ramberg–Osgood material law were also considered. A highlight of Lazzarin's derivations is a definite relationship between elastic and plastic NSIFs. A uniform analysis of nonlinear notch stress fields was presented based on the total-strain power-law in comparison to Neuber's different analytical approach. The well-known Neuber rule, establishing a simple relationship between the stress and strain concentration factors at elastic-plastically deformed notches, deviates by a factor of 1.0–2.0 (largest for perfectly-plastic materials) from Lazzarin's solution.

The second area of notch mechanics elaborated by Paolo Lazzarin from 2001 onward is the local strain energy density (SED) concept. Since the NSIFs represent odd singularities which depend on the notch opening angle, a comparison between the fatigue limits of different weld geometries can only be carried out by using the SED averaged over a small control volume surrounding the point of stress singularity. The averaged SED is always bounded, independent of the notch acuity. The radius of the control volume is understood as a material property which can be determined on a statistical basis evaluating a large body of fatigue test data related to fillet-welded joints of various geometries and dimensions. Fatigue strength values in terms of averaged SED amplitudes dependent on number of loading cycles were presented in the form of a scatterband with a width and gradient well in agreement with comparable nominal stress relationships in the codes.

The agreement of the parameters just mentioned is resulting from the fact that, in many cases of practical interest, not only the fatigue crack initiation life is correlated with the local SED value but also the total life, provided the major part of the fatigue life can be assigned to microcrack initiation and propagation inside the zone governed by the notch stress singularity.

The SED approach for the fatigue assessment of non-codified welded joints, in comparison to alternative local approaches such as NSIF, FNR or  $J$ -integral, has the advantage that the local SED can easily be determined from linear-elastic FE models with coarse meshing without major loss in accuracy. The reason is that the nodal point displacements determine directly the SED and not the displacement derivatives. Any FE analyst in industry will highly appreciate this feature.

A SED-based version of the Atzori–Lazzarin diagram correlating notch sensitivity with defect sensitivity related to fatigue loading has been derived.

Paolo Lazzarin has also successfully applied the local SED concept to brittle fractures under monotonic loading, considering especially acrylic glass specimens. The concept was also extended into the elastic-plastic range, to multiaxial fatigue and to blunt notches.

The link between the locally averaged SED and Rice's  $J$ -integral has been established both for pointed and blunt notches. Lazzarin's last finding in this area was that the  $J$ -integrals at the pointed V-notch tip are not identical when modelling the material behaviour by the total-strain power-law and the Ramberg–Osgood relationship in comparison. His early death did not allow him to pursue this discrepancy further.

The third area of notch mechanics, to which Paolo Lazzarin contributed substantially, is Neuber's concept of fictitious notch rounding (FNR) combined with Radaj's application to welded joints within a worst case scenario. Lazzarin's idea to use the average linear-elastic SED in a control volume as relevant for fatigue or brittle fracture was stimulated by Neuber's earlier concept to average the linear-elastic stresses over a microstructural support length in the direction of crack initiation. A first comparison between the fatigue limits of typical welded joints estimated according to the two concepts in 2007 was encouraging. In the years thereafter, substantial improvements and extensions of the FNR approach have been achieved.

The support factor correlating the fictitious notch radius to the microstructural support length with consideration of the multiaxiality conditions was determined for the in-plane tension and out-of-plane shear loading modes depending on notch opening angle and multiaxial failure criterion. This denotes substantial progress in respect of application to welded joints, because the notch opening angle is a decisive parameter here whereas Neuber has neglected this influence. In a further effort, in-plane shear loading was included which necessitates consideration of out-of-bisector crack initiation. Paolo Lazzarin's last article in the field of notch mechanics was related to the FNR concept applied to V-notches under

mixed-mode in-plane loading conditions. The theoretical derivations and mathematical formulations in this article are of unsurpassable elegance without neglect of the application aspects. They carry on the high quality standard in Neuber's famous book on notch mechanics.



Figure 1: Photo of Prof. Lazzarin

Paolo Lazzarin's contributions to theoretical and applied mechanics are of lasting value. They are an indispensable reference for the scientists after him.

Paolo Lazzarin's scientific achievements were based not only on a profound understanding of continuum mechanics combined with an exceptional talent in applied mathematics, but also on his likeable character. The foremost character trait was warm-heartedness, and his generosity was unsurpassable. He paid full respect to any scientific colleague or disciple, not forgetting emotional uprising where it was justified. Scientific questions were always answered without delay and in detail. He was a master in didactic clearness and never forgot to emphasize the historical originators of a concept, a formula or an idea. He has guided many students and young researchers through the confusing multitude of scientific methods and approaches. He shared ideas and knowledge with them as a true friend.

### SOME EXPRESSIONS FOR SED IN THE CONTROL VOLUME

With the aim of clarifying the base of the final synthesis carried out in this paper, this section summarises the analytical frame of SED approach.

With reference to the polar coordinate system shown in Fig. 2, with the origin located at point O, mode I stress distribution ahead of a V-notch tip is given by the following expressions [65]:

$$\sigma_{ij} = a_1 r^{\lambda_1 - 1} \left[ f_{ij}(\theta, \alpha) + \left( \frac{r}{r_0} \right)^{\mu_1 - \lambda_1} g_{ij}(\theta, \alpha) \right] \quad (1)$$

where  $\lambda_1 > \mu_1$  and the parameter  $a_1$  can be expressed either via the notch stress intensity factor  $K_1$  in the case of a sharp, zero radius, V-notch or by means of the elastic maximum notch stress  $\sigma_{tip}$  in the case of blunt V-notches. In Eq. (1)  $r_0$  is the distance evaluated on the notch bisector line between the V-notch tip and the origin of the local coordinate system;  $r_0$  depends both on the notch root radius  $R$  and the opening angle  $2\alpha$  (Fig. 2), according to the expression  $r_0 = R[(\pi - 2\alpha)/(2\pi - 2\alpha)]$ . The angular functions  $f_{ij}$  and  $g_{ij}$  are given in Ref. [65]:



$$\begin{Bmatrix} f_{\theta\theta} \\ f_{rr} \\ f_{r\theta} \end{Bmatrix} = \frac{1}{1 + \lambda_1 + \chi_{b_1}(1 - \lambda_1)} \left[ \begin{Bmatrix} (1 + \lambda_1) \cos(1 - \lambda_1)\theta \\ (3 - \lambda_1) \cos(1 - \lambda_1)\theta \\ (1 - \lambda_1) \sin(1 - \lambda_1)\theta \end{Bmatrix} + \chi_{b_1}(1 - \lambda_1) \begin{Bmatrix} \cos(1 + \lambda_1)\theta \\ -\cos(1 + \lambda_1)\theta \\ \sin(1 + \lambda_1)\theta \end{Bmatrix} \right] \quad (2)$$

$$\begin{Bmatrix} g_{\theta\theta} \\ g_{rr} \\ g_{r\theta} \end{Bmatrix} = \frac{q}{4(q - 1)[1 + \lambda_1 + \chi_{b_1}(1 - \lambda_1)]} \left( \chi_{d_1} \begin{Bmatrix} (1 + \mu_1) \cos(1 - \mu_1)\theta \\ (3 - \mu_1) \cos(1 - \mu_1)\theta \\ (1 - \mu_1) \sin(1 - \mu_1)\theta \end{Bmatrix} + \chi_{e_1} \begin{Bmatrix} \cos(1 + \mu_1)\theta \\ -\cos(1 + \mu_1)\theta \\ \sin(1 + \mu_1)\theta \end{Bmatrix} \right) \quad (3)$$

The eigenfunctions  $f_{ij}$  depend only on Williams' eigenvalue,  $\lambda_1$ , which controls the sharp solution for zero notch radius [66]. The eigenfunctions  $g_{ij}$  mainly depend on eigenvalue  $\mu_1$ , but are not independent from  $\lambda_1$ . Since  $\mu_1 < \lambda_1$ , the contribution of  $\mu$ -based terms in Eq. (1) rapidly decreases with the increase of the distance from the notch tip. All parameters in Eqs.(2,3) have closed form expressions but here, for the sake of brevity, only some values for the most common angles are reported in [65].

Under the plane strain conditions, the eigenfunctions  $f_{ij}$  and  $g_{ij}$  satisfy the following expressions:

$$f_{\bar{x}\bar{x}}(\theta) = \nu(f_{\theta\theta}(\theta) + f_{rr}(\theta)) \quad g_{\bar{x}\bar{x}}(\theta) = \nu(g_{\theta\theta}(\theta) + g_{rr}(\theta)) \quad (4)$$

whereas  $f_{zz}(\theta) = g_{zz}(\theta) = 0$  under plane stress conditions.

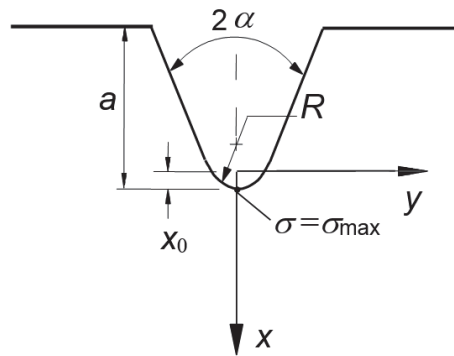


Figure 2: Notch geometry and coordinate system

The SED approach is based on the idea that under tensile stresses failure occurs when  $\bar{W} = W_c$ , where the critical value  $W_c$  obviously varies from material to material. If the material behaviour is ideally brittle, then  $W_c$  can be evaluated by using simply the conventional ultimate tensile strength  $\sigma_t$ , so that  $W_c = \sigma_t^2 / 2E$ .

Often un-notched specimens exhibit a non-linear behaviour whereas the behaviour of notched specimens remains linear. Under these circumstances the stress  $\sigma_t$  should be substituted by “the maximum normal stress existing at the edge at the moment preceding the cracking”, as underlined in Ref. [21] where it is also recommended to use tensile specimens with semicircular notches.

In plane problems, the control volume becomes a circle or a circular sector with a radius  $R_0$  in the case of cracks or pointed V-notches in mode I or mixed, I+II, mode loading (Fig. 3a,b). Under plane strain conditions, a useful expression for  $R_0$  has been provided considering the crack case [41]:

$$R_0 = \frac{(1 + \nu)(5 - 8\nu)}{4\pi} \left( \frac{K_{IC}}{\sigma_t} \right)^2 \quad (5)$$

If the critical value of the NSIF is determined by means of specimens with  $2\alpha \neq 0$ , the critical radius can be estimated by means of the expression:

$$R_0 = \left[ \frac{I_1 \times K_{1C}^2}{4\lambda_1 (\pi - \alpha) E W_c} \right]^{1/(2-2\lambda_1)} \quad (6)$$

When  $2\alpha=0$ ,  $K_{1C}$  equals the fracture toughness  $K_{IC}$ .

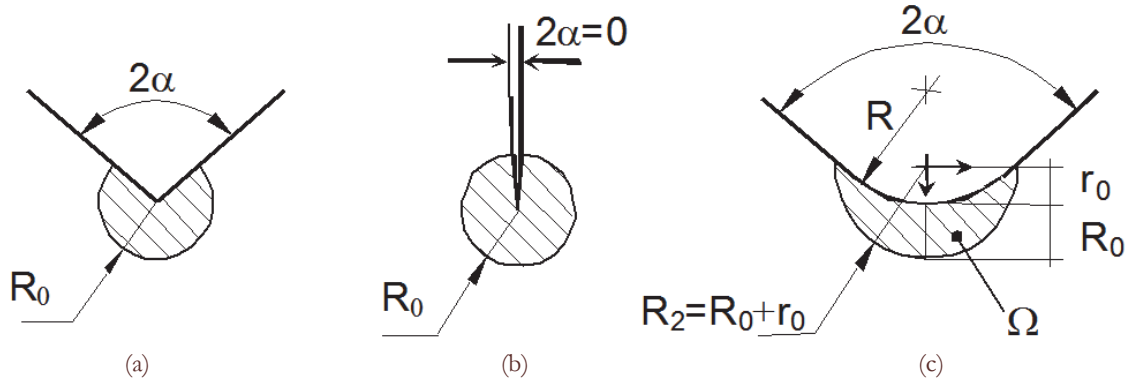


Figure 3: Critical volume (area) for sharp V-notch (a), crack (b) and blunt V-notch (c) under mode I loading. Distance  $r_0 = R \times (\pi - 2\alpha) / (2\pi - 2\alpha)$ .

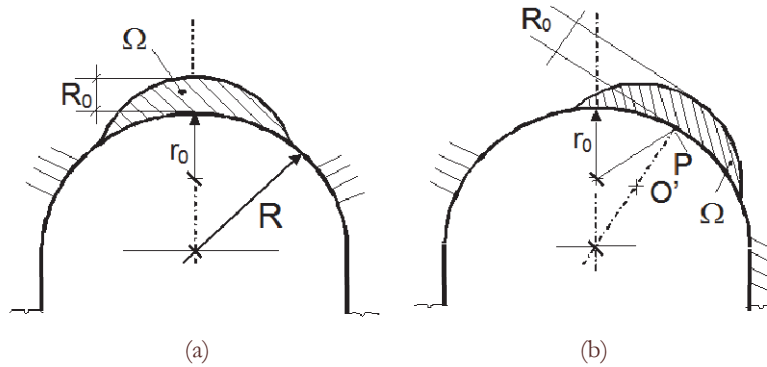


Figure 4: Critical volume for U-notch under mode I (a) and mixed mode loading (b). Distance  $r_0 = R / 2$  according to Refs. [15] and [67].

In the case of blunt notches, the area assumes a crescent shape, with  $R_0$  being its maximum width as measured along the notch bisector line (Fig. 2c) [11]. Under mixed-mode loading, the control area is no longer centred with respect to the notch bisector, but rigidly rotated with respect to it and centred on the point where the maximum principal stress reaches its maximum value [58, 59]. This rotation is shown in Figure 4 where the control area is drawn for a U-shaped notch both under mode I loading (Fig. 4a) and mixed-mode loading (Fig. 4b).

The parameter  $a_1$  of Eq. (1) can be linked to the mode I notch stress intensity factor by means of the simple expression

$$a_1 = \frac{K_1}{\sqrt{2\pi}} \quad (7)$$

where  $K_1$  assumes the following form according to the definition given in Ref.[ 68]:

$$K_1 = \sqrt{2\pi} \lim_{r \rightarrow 0} [\sigma_\theta(r, \theta = 0)] r^{1-\lambda_1} \quad (8)$$





In the presence of a notch root radius equal to zero, the distance  $r_0$  is also zero, and all  $\mu$ -related terms in Eq. (1) disappear. It is possible to determine the total strain energy over the area of radius  $R_0$  and then the mean value of the elastic SED referred to the area  $\Omega$ . The final relationship is:

$$\bar{W}_1 = \frac{I_1}{4 E \lambda_1 (\pi - \alpha)} \left( \frac{K_1}{R_0^{1-\lambda_1}} \right)^2 \quad (9)$$

where  $\lambda_1$  is Williams' eigenvalue and  $K_1$  the mode I notch stress intensity factor. The parameter  $I_1$  is different under plane stress and plane strain conditions for different values of the Poisson's ratio  $\nu$  [60]. Equation (9) was extended to pointed V-notches in mixed, I+II, mode [7] as well as to cases where mode I loads were combined with mode III loads [9]. It is important to underline the influence of the Poisson's ratio on the  $I_1$  values in the case of sharp notches. For a notch opening angle smaller than 60 degrees,  $I_1$  varies strongly from  $\nu=0.1$  to  $\nu=0.4$ . This fact confirms the important and not negligible effect of the Poisson's ratio while discussing sharp or quasi-sharp notches in agreement with the effect of this parameter on the kind of singularities, weak or strong, highlighted in Ref. [1] in the case of a constrained micro-notch at the front of a free edge macro-crack.

In the presence of rounded V-notches it is possible to link the parameter  $a_1$  of Eq. (1) to the maximum principal stress present at the notch tip:

$$a_1 = \frac{\sigma_{tip} r_0^{1-\lambda_1}}{1 + \tilde{\omega}_1} \quad (10)$$

where  $\tilde{\omega}_1$  is the parameter already listed in the original paper. By using the elastic maximum notch stress, it is possible to determine the total strain energy over the area  $\Omega$  and then the mean value of the SED. When the area embraces the semicircular edge of the notch (and not its rectilinear flanks), the mean value of SED can be expressed in the following form [11]:

$$\bar{W}_1 = F(2\alpha) \times H(2\alpha, \frac{R_0}{R}) \times \frac{\sigma_{tip}^2}{E} \quad (11)$$

where  $F(2\alpha)$  depend on previously defined parameters

$$F(2\alpha) = \left( \frac{q-1}{q} \right)^{2(1-\lambda_1)} \left[ \frac{\sqrt{2\pi}}{1 + \tilde{\omega}_1} \right]^2 \quad (12)$$

By simply using the definition of the mode I NSIF for blunt V-notches [69] a simple relationship between  $\sigma_{tip}$  and  $K_{R,I}^V$  can be obtained as follows:

$$K_{R,I}^V = \sqrt{2\pi} \frac{\sigma_{tip}}{1 + \tilde{\omega}_1} \left( \frac{q-1}{q} R \right)^{1-\lambda_1} = \sqrt{F(2\alpha)} \sigma_{tip} R^{1-\lambda_1} \quad (13)$$

Then it is possible to rewrite Eq. (11) in a more compact form:

$$\bar{W}_1 = H(2\alpha, \frac{R_0}{R}) \times \frac{(K_{R,I}^V)^2}{E} \times \frac{1}{R_0^{2(1-\lambda_1)}} \quad (14)$$

Equation (14) will be used to summarise all results from blunt notches (U- and V-notches) subjected to mode I loading.



As opposed to the direct evaluation of the NSIFs, which needs very refined meshes, the mean value of the elastic SED on the control volume can be determined with high accuracy by using coarse meshes [70, 71]. Very refined meshes are necessary to directly determined the NSIFs from the local stress distributions. Refined meshes are not necessary when the aim of the finite element analysis is to determine the mean value of the local strain energy density on a control volume surrounding the points of stress singularity. The SED in fact can be derived directly from nodal displacements, so that also coarse meshes are able to give sufficiently accurate values for it. Some recent contributions document the weak variability of the SED as determined from very refined meshes and coarse meshes, considering some typical welded joint geometries and provide a theoretical justification to the weak dependence exhibited by the mean value of the local SED when evaluated over a control volume centred at the weld root or the weld toe. On the contrary singular stress distributions are strongly mesh dependent. The NSIFs can be estimated from the local SED value of pointed V-notches in plates subjected to mode I, Mode II or a mixed mode loading. Taking advantage of some closed-form relationships linking the local stress distributions ahead of the notch to the maximum elastic stresses at the notch tip the coarse mesh SED-based procedure is used to estimate the relevant theoretical stress concentration factor  $K_t$  for blunt notches considering, in particular, a circular hole and a U-shaped notch, the former in mode I loading, the latter also in mixed, I + II, mode [70-71]. Other important advantages can be achieved by using the SED approach. The most important are as follows:

- It permits consideration of the scale effect which is fully included in the Notch Stress Intensity Factor Approach
  - It permits consideration of the contribution of different Modes.
  - It permits consideration of the cycle nominal load ratio.
  - It overcomes the complex problem tied to the different NSIF units of measure in the case of different notch opening angles (i.e crack initiation at the toe ( $2\alpha=135^\circ$ ) or root ( $2\alpha=0^\circ$ ) in a welded joint)
  - It overcomes the complex problem of multiple crack initiation and their interaction on different planes.
  - It directly takes into account the T-stress and this aspect becomes fundamental when thin structures are analysed [72].
  - It directly includes three-dimensional effects and out-of-plane singularities not assessed by Williams' theory [73-77].
- The mean value of the strain energy density (SED) in a circular sector of radius  $R_0$  located at the fatigue crack initiation sites has been used to summarise fatigue strength data from steel welded joints of complex geometry (Fig. 4).

## SYNTHESIS BASED ON SED IN A CONTROL VOLUME

Local strain energy density  $\Delta\bar{W}$  averaged in a finite size volume surrounding weld toes and roots is a scalar quantity which can be given as a function of mode I-II NSIFs in plane problems [8] and mode I-II-III NSIFs in three dimensional problems [9]. The evaluation of the local strain energy density needs precise information about the control volume size. From a theoretical point of view the material properties in the vicinity of the weld toes and the weld roots depend on a number of parameters as residual stresses and distortions, heterogeneous metallurgical micro-structures, weld thermal cycles, heat source characteristics, load histories and so on. To devise a model capable of predicting  $R_0$  and fatigue life of welded components on the basis of all these parameters is really a task too complex. Thus, the spirit of the approach is to give a simplified method able to summarise the fatigue life of components only on the basis of geometrical information, treating all the other effects only in statistical terms, with reference to a well-defined group of welded materials and, for the time being, to arc welding processes.

In a plane problem all stress and strain components in the highly stressed region are correlated to mode I and mode II NSIFs. Under a plane strain hypothesis, the strain energy included in a semicircular sector shown in Figure 2 is [7, 13]

$$\Delta\bar{W} = \left\{ \frac{e_1}{E} \left[ \frac{\Delta K_1^N}{R_0^{1-\lambda_1}} \right]^2 + \frac{e_2}{E} \left[ \frac{\Delta K_2^N}{R_0^{1-\lambda_2}} \right]^2 + \frac{e_3}{E} \left[ \frac{\Delta K_3^N}{R_0^{1-\lambda_3}} \right]^2 \right\} \quad (15)$$

where  $R_0$  is the radius of the semicircular sector and  $e_1$ ,  $e_2$  and  $e_3$  are functions that depend on the opening angle  $2\alpha$  and the Poisson ratio  $\nu$ .

The material parameter  $R_0$  can be estimated by using the fatigue strength  $\Delta\sigma_A$  of the butt ground welded joints (in order to quantify the influence of the welding process, in the absence of any stress concentration effect) and the NSIF-based fatigue strength of welded joints having a V-notch angle at the weld toe constant and large enough to ensure the non singularity of mode II stress distributions.



A convenient expression is [7]:

$$R_0 = \left( \frac{\sqrt{2e_1} \Delta K_{1,A}^N}{\Delta \sigma_A} \right)^{\frac{1}{1-\lambda_1}} \quad (16)$$

where both  $\lambda_1$  and  $e_1$  depend on the V-notch angle. Eq. (16) makes it possible to estimate the  $R_0$  value as soon as  $\Delta K_{1,A}^N$  and  $\Delta \sigma_A$  are known. At  $N_A = 5 \cdot 10^6$  cycles and in the presence of a nominal load ratio  $\tilde{R}$  equal to zero a mean value  $\Delta K_{1,A}^N$  equal to 211 MPa mm<sup>0.326</sup> was found re-analysing experimental results taken from the literature [10]. For butt ground welds made of ferritic steels a mean value  $\Delta \sigma_A = 155$  MPa (at  $N_A = 5 \cdot 10^6$  cycles) was found [79]. Then, by introducing the above mentioned value into Eq.(16), one obtains for steel welded joints with failures from the weld toe  $R_0 = 0.28$  mm. It is interesting to learn that, for welded joints made of structural steels, different expressions for  $\Delta K_{th}$  taken from the literature were reported in Ref. [79], from which  $\Delta K_{th} = 180 \text{ MPa} \sqrt{\text{mm}}$  (5.7 MPa m<sup>1/2</sup>). In the case  $2\alpha=0$  and fatigue crack initiation at the weld root Eq.(16) gives  $R_0=0.36$  mm, by neglecting the mode II contribution and using  $e_1=0.133$ , Eq.(7),  $\Delta K_{1,A}^N = 180 \text{ MPa mm}^{0.5}$ , and, once again,  $\Delta \sigma_A = 155$  MPa. This means that the choice to use a critical radius equal to 0.28 mm both for toe and root failures is a sound engineering approximation. By modelling the weld toe regions as sharp V-notches and using the local strain energy, more than 900 fatigue strength data from welded joints with weld toe failure were analysed and the first theoretical scatter band in terms of SED was obtained [8]. The geometry exhibited a strong variability of the main plate thickness (from 6 to 100 mm), the transverse plate (from 3 to 200 mm) and the bead flank (from 110 to 150 degrees).

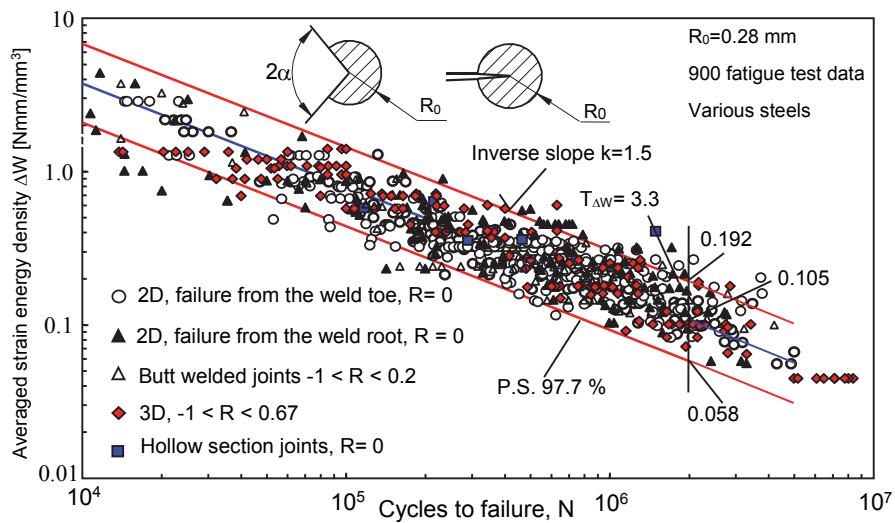


Figure 5: Fatigue strength of welded joints as a function of the averaged local strain energy density;  $\tilde{R}$  is the nominal load ratio.

The synthesis of all those data is shown in Figure 5, where the number of cycles to failure is given as a function of  $\Delta \bar{W}_1$  (the Mode II stress distribution being non singular for all those geometries). The figure includes data obtained both under tension and bending loads, as well as from “as-welded” and “stress-relieved” joints. The scatter index  $T_W$ , related to the two curves with probabilities of survival  $P_S = 2.3\%$  and  $97.7\%$ , is 3.3, to be compared with the variation of the strain energy density range, from about 4.0 to about 0.1 MJ/m<sup>3</sup>.  $T_W = 3.3$  becomes equal to 1.50 when reconverted to an equivalent local stress range with probabilities of survival  $P_S = 10\%$  and  $90\%$  ( $T_\sigma = \sqrt{3.3} / 1.21 = 1.5$ ). The scatterband proposed was latter applied in [10] to a larger bulk of experimental data, which included also fatigue failures from the weld root.

Dealing with static loading, the local SED values are normalised to the critical SED values (as determined from unnotched specimens) and plotted as a function of the  $R/R_0$  ratio. The data related to the experimental program of PMMA tested at  $-60^\circ\text{C}$  [56-59] are summarised together with other data taken from a data base due related to PMMA tested at room

temperature [45, 54-55]. The final synthesis has been carried out by normalising the local SED to the critical SED values (as determined from unnotched, plain specimens) and plotting this non-dimensional parameter as a function of the  $R/R_0$  ratio. A scatterband is obtained whose mean value does not depend on  $R/R_0$ , whereas the ratio between the upper and the lower limits are found to be about equal to  $1.3/0.8=1.6$  (Figure 6). The strong variability of the non-dimensional radius  $R/R_0$  (notch root radius to control volume radius ratio, ranging here from about zero to about 500) makes stringent the check of the approach based on the local SED. The scatterband presented contains failure data from 20 different ceramics, 4 PVC foams and some metallic materials.

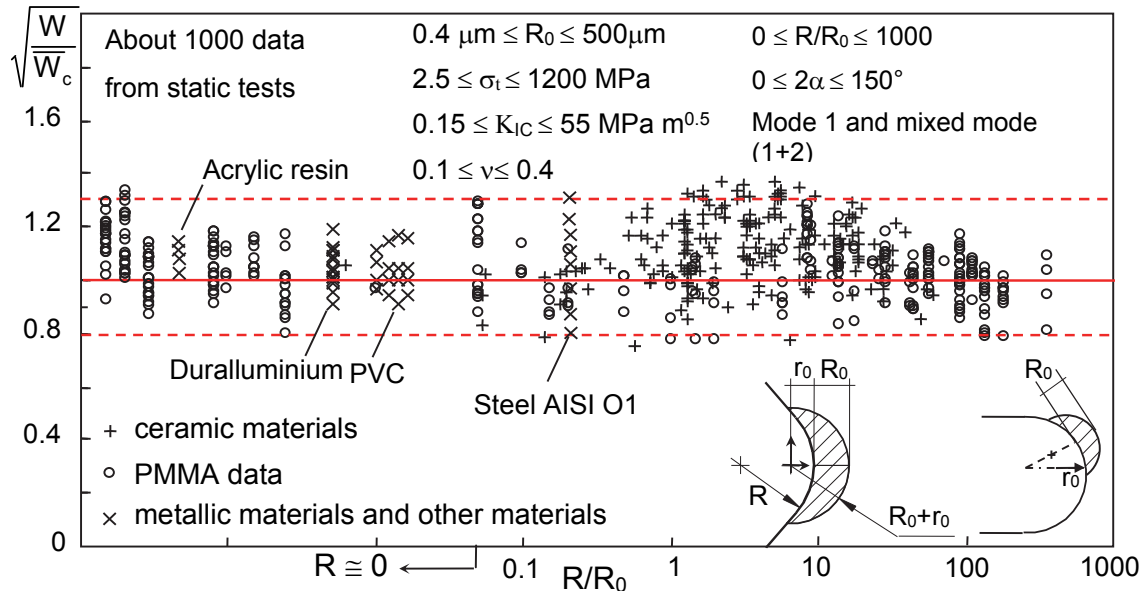


Figure 6: Synthesis of data taken from the literature.

## CONCLUSIONS

For many years the Strain Energy Density (SED) has been used to formulate failure criteria for materials exhibiting both ductile and brittle behaviour. SED is the most fundamental quantity in Mechanics being all physical quantities expressible in terms of it. From pico to macroscopic scale the energy absorption and dissipation can explain the most complex phenomena tied to fracture initiation and propagation. Keeping in mind that the design rules valid for large bodies (i.e high volume to surface ratio) can not be directly translated and applied to small bodies where local inhomogeneities play a fundamental role for the material damage initiation and propagation and being also aware of the recent contributions and efforts to develop a multiscaling and segmentation scheme able to capture the complex phenomena that happen at every level from pico to macro, the main purpose of the paper is to present a review of the approach based on the mean value of the local strain energy density. Dealing with static loading the approach is applied here to different materials and geometries both, under mode I and mixed mode (I+II) loading. About one thousand experimental data, taken from the recent literature, are involved in the synthesis. They were from U- and V-notched specimens made of very different materials. A scatter band is proposed by using as a synthesis parameter the value of the local energy averaged over control volume (of radius  $R_0$ ), normalised by the critical energy of the material. Such a normalised energy is plotted as a function of the notch radius to critical radius ratio,  $R/R_0$ . The strain energy density (SED) in a circular sector of radius  $R_0$  located at the crack initiation sites has successfully been used to summarise also about nine hundred data from fatigue failures of welded joints. Under the hypothesis that all material inhomogeneities can be averaged, that ceases to be valid at pico- and micro-levels but at the same time is the basis of the volume-based theories applied to structural components, the Strain Energy Density Approach is shown to be a powerful tool both for static and fatigue strength assessment of notched and welded structures.



## REFERENCES

- [1] Tang, X.S., Sih, G.C., Weak and strong singularities reflecting multiscale damage: Micro-boundary conditions for free-free, fixed-fixed and free-fixed constraints, *Theor. Appl. Fract. Mech.*, 43(2005) 5-62. DOI: 10.1016/j.tafmec.2004.12.002.
- [2] Sih, G.C. Tang, X.S., Scaling of volume energy density function reflecting damage by singularities at macro-, meso- and microscopic level, *Theor. Appl. Fract. Mech.*, 43 (2005) 211-231. DOI: 10.1016/j.tafmec.2005.01.006.
- [3] Sih, G.C., Multiscaling in molecular and continuum mechanics: interaction of time and size from macro to nano, Springer, Dordrecht, (2007). DOI: 10.1007/978-1-4020-5062-6.
- [4] Sih, G.C., Crack tip mechanics based on progressive damage of arrow: Hierarchy of singularities and multiscale segments, *Theor. Appl. Fract. Mech.*, 51 (2009) 1-32. DOI: 10.1016/j.tafmec.2009.01.007.
- [5] Sih, G.C. Ideomechanics of transitory and dissipative systems associated with length, velocity, mass and energy, *Theor. Appl. Fract. Mech.*, 51(2009) 149-160. DOI: 10.1016/j.tafmec.2009.05.008.
- [6] Sih, G.C. Energy absorption and dissipation associated with mass activation and deactivation for open systems, *Theor. Appl. Fract. Mech.*, 52 (2009), 63-71. DOI: 10.1016/j.tafmec.2009.08.008.
- [7] Lazzarin, P. Zambardi, R., A finite-volume-energy based approach to predict the static and fatigue behaviour of components with sharp V-shaped notches, *Int. J. Fracture*, 112 (2001) 275-298. DOI: 10.1023/A:1013595930617.
- [8] Lazzarin, P., Lassen, T., Livieri, P., A Notch Stress Intensity approach applied to fatigue life predictions of welded joints with different local toe geometry, *Fatigue Fract. Eng. Mater. Struct.*, 26 (2003) 49-58. DOI: 10.1046/j.1460-2695.2003.00586.x.
- [9] Lazzarin, P., Sonsino, C.M., Zambardi, R. A notch stress intensity approach to assess the multiaxial fatigue strength of welded tube-to flange joints subjected to combined loadings, *Fatigue Fract. Eng. Mater. Struct.*, 27 (2004) 127-140. DOI: 10.1111/j.1460-2695.2004.00733.x.
- [10] Livieri, P., Lazzarin, P., Fatigue strength of steel and aluminium welded joints based on generalised stress intensity factors and local strain energy values, *Int. J. Fracture*, 133 (2005) 247-278. DOI: 10.1007/s10704-005-4043-3.
- [11] Lazzarin, P., Berto F., Some expressions for the strain energy in a finite volume surrounding the root of blunt V-notches. *Int. J. Fracture*, 135 (2005) 161-185. DOI: 10.1007/s10704-005-3943-6.
- [12] Lazzarin, P., Berto F., From Neuber's elementary volume to Kitagawa and Atzori's diagrams: an interpretation based on local energy, *Int. J. Fracture*, 135 (2005) L33-L38. DOI: 10.1007/s10704-005-4393-x.
- [13] Lazzarin, P., Livieri, P., Berto, F., Zappalorto, M., Local strain energy density and fatigue strength of welded joints under uniaxial and multiaxial loading, *Eng. Fract. Mech.*, 75 (2008) 1875-1889. DOI: 10.1016/j.engfracmech.2006.10.019.
- [14] Lazzarin, P., Berto, F., Control volumes and strain energy density under small and large scale yielding due to tensile and torsion loading, *Fatigue Fract. Eng. Mater. Struct.*, 31 (2008) 95-107. DOI: 10.1111/j.1460-2695.2007.01206.x.
- [15] Neuber, H., *Kerbspannungslehre*, 2nd edn, Springer-Verlag, Berlin, (1958).
- [16] Neuber, H., Über die Berücksichtigung der Spannungskonzentration bei Festigkeitsberechnungen. *Konstruktion*, 20 (1968) 245-251.
- [17] Neuber, H., *Kerbspannungslehre*, 3rd edn, Springer-Verlag, Berlin, (1985).
- [18] McClintock, F.A., Ductile fracture instability in shear, *J. Appl. Mech.*, 25 (1958) 582-588.
- [19] Ritchie, R.O., Knott, J., Rice, J.R., On the relation between critical tensile stress in fracture toughness in mild steel, *J. Mech. Phys. Sol.*, 21 (1973) 395-410. DOI: 10.1016/0022-5096(73)90008-2.
- [20] Knésl, Z., A criterion of V-notch stability, *Int. J. Fracture*, 48 (1991) R79-R83. DOI: 10.1007/BF00012922.
- [21] Seweryn, A., Brittle fracture criterion for structures with sharp notches, *Eng. Fract. Mech.*, 47(1994) 673-681. DOI: 10.1016/0013-7944(94)90158-9.
- [22] Novozhilov, V.V., On necessary and sufficient criterion of brittle fracture. *Prikladnaja Matematika i Mechanika*, 33 (1969) 212-222.
- [23] Seweryn, A., Mróz, Z., A non-local stress failure condition for structural elements under multiaxial loading. *Eng. Fract. Mech.*, 51(1995) 955-973. DOI: 10.1016/0013-7944(94)00335-F.
- [24] Seweryn, A. Poskrobko, S., Mróz, Z., Brittle fracture in plane elements with sharp notches under mixed-mode loading. *J. Eng. Mech.*, 123 (1997) 535-543. DOI: 10.1061/(ASCE)0733-9399(1997)123:6(535).
- [25] Sheppard, S.D., Field effects in fatigue crack initiation: long life fatigue strength. *Trans. ASME. J. Mech. Des.*, 113 (1991) 188-194. Doi:10.1115/1.2912768.
- [26] Beltrami, E., Sulle condizioni di resistenza dei corpi elastici, *Rend. R. Ist. Lombardo di Scienze, Lettere e Arti*, 18 (1885) 704 (in Italian). Doi: 10.1007/BF02824697.





- [27] Sih, G.C., A special theory of crack propagation: methods of analysis and solutions of crack problems, *Mechanics of Fracture*, edited by GC Sih Noordhoff, International Publishing, Leyden, (1973).
- [28] Sih, G.C., Strain-energy-density factor applied to mixed mode crack problems. *Int. J. Fracture*, 10 (1974) 305-321. DOI: 10.1007/BF00035493.
- [29] Sih, G.C., Surface layer energy and strain energy density for a blunted crack or notch, *Prospect of Fracture Mechanics*, edited by GC Sih, H.C. van Elst and D. Broek, Noordhoff International Publishing, Leyden, (1974).
- [30] Sih, G.C., *Mechanics of Fracture Initiation and Propagation: Surface and volume energy density applied as failure criterion*, Kluwer Academic Publisher, Dordrecht, (1991). DOI: 10.1007/978-94-011-3734-8.
- [31] Sih, G.C., Ho, J.W., Sharp notch fracture strength characterized by critical energy density, *Theor. Appl. Fract. Mech.*, 16 (1991) 179-214. DOI: 10.1016/0167-8442(91)90044-K.
- [32] Sih, G.C., Czoboly, E., Gillemot, L.F., Absorbed Specific Energy and Strain Energy Density Criterion, *Proceedings of an International Symposium on Absorbed Specific Energy and Strain Energy Density Criterion*, Budapest, September 1980. In memory of the late Professor László Gillemot, Martinus Nijhoff Publishers, (1982), The Netherlands.
- [33] Gillemot, L.F., Brittle fracture of welded materials, In: *Commonwealth Welding Conference C.7.* (1965) 353-358.
- [34] Gillemot, L.F., Criterion of crack initiation and spreading. *Eng. Fract. Mech.*, 8 (1976) 239-253. DOI: 10.1016/0013-7944(76)90089-8.
- [35] Gillemot, L.F., Czoboly, E., Havas, I., Fracture mechanics applications of absorbed specific fracture energy: notch and unnotched specimens, *Theor. Appl. Fract. Mech.*, 4 (1985) 39-45. DOI: 10.1016/0167-8442(85)90041-2.
- [36] Gdoutos, E.E., *Fracture Mechanics Criteria and Applications*, Kluwer Academic Publishers, Dordrecht, Boston, London (1990). DOI: 10.1007/978-94-009-1956-3.
- [37] Glinka, G., Molski, K., A method of elastic-plastic stress and strain calculation at a notch root, *Mater. Sci. Eng.*, 50 (1981) 93-100. DOI: 10.1016/0025-5416(81)90089-6.
- [38] Glinka, G., Energy density approach to calculation of inelastic strain-stress near notches and cracks. *Eng. Fract. Mech.*, 22 (1985) 485-508. DOI:10.1016/0013-7944(85)90148-1.
- [39] Ellyin, F., *Fatigue Damage, Crack Growth and Life Prediction*, Chapman & Hall, London, (1997).
- [40] Ellyin, F., Kujawski, D. Generalization of notch analysis and its extension to cyclic loading, *Eng. Fract. Mech.*, 32 (1989) 819-826. DOI: 10.1016/0013-7944(89)90178-1.
- [41] Yosibash, Z., Bussiba, Ar., Gilad, I., Failure criteria for brittle elastic materials. *Int. J. Fracture*, 125 (2004) 307-333. DOI: 10.1023/B:FRAC.0000022244.31825.3b.
- [42] Kipp, M. E., Sih, G.C., The strain energy density failure criterion applied to notched elastic solids, *Int. J. Solids Struct.*, 11(1975) 153–173. DOI: 10.1016/0020-7683(75)90050-5.
- [43] Carpinteri, A., Stress singularity and generalised fracture toughness at the vertex of re-entrant corners, *Eng. Fract. Mech.*, 26 (1987) 143-155. DOI: 10.1016/0013-7944(87)90086-5.
- [44] Nui, L.S., Chehimi, C. Pluvinage, G., Stress field near a large blunted tip V-Notch and application of the concept of the critical notch stress intensity factor (NSIF) to the fracture toughness of very brittle materials, *Eng. Fract. Mech.*, 49 (1994) 325-335. DOI: 10.1016/0013-7944(94)90262-3.
- [45] Gómez, F.J., Elices, M., Valiente, A. Cracking in PMMA containing U-shaped notches, *Fatigue Fract. Eng. Mater. Struct.*, 23 (2000) 795-803. DOI: 10.1046/j.1460-2695.2000.00264.x.
- [46] Atzori, B., Lazzarin, P. Notch sensitivity and defect sensitivity: two sides of the same medal, *Int. J. Fracture*, 107 (2001) L3-L8.
- [47] Strandberg, M., Fracture at V-notches with contained plasticity, *Eng. Fract. Mech.*, 69 (2002) 403-415. DOI: 10.1016/S0013-7944(01)00079-0.
- [48] Atzori, B., Lazzarin, P., Meneghetti, G., Fracture mechanics and notch sensitivity, *Fatigue Fract. Eng. Mater. Struct.*, 26 (2003) 257-267. DOI: 10.1046/j.1460-2695.2003.00633.x.
- [49] Gogotsi, G.A., Fracture toughness of ceramics and ceramic composites, *Ceramics Int.*, 7 (2003) 777-884. DOI: 10.1016/S0272-8842(02)00230-4.
- [50] Leguillon, D., Yosibash, Z., Crack onset at a V-notch. Influence of the notch tip radius, *Int. J. Fract.*, 122 (2003)1-21. DOI: 10.1023/B:FRAC.0000005372.68959.
- [51] Dini, D., Hills, D., Asymptotic characterisation of nearly-sharp notch root stress fields, *Int. J. Fracture*, 130 (2004) 651-666. DOI: 10.1007/s10704-004-2510-x.
- [52] Yosibash, Z., Priel, E. Leguillon, D., A failure criterion for brittle elastic materials under mixed-mode loading, *Int. J. Fracture*, 141 (2006) 291-312. DOI: 10.1007/s10704-006-0083-6.



- [53] Leguillon, D., Quesada, D. Putot, C., Martin, E., Prediction of crack initiation at blunt notches and cavities – size effects, *Eng. Fract. Mech.* 74 (2007) 2420-2436. DOI:10.1016/j.engfracmech.2006.11.008.
- [54] Gómez, F.J., Elices, M., Fracture of components with V-shaped notches. *Eng. Fract. Mech.*, 70 (2003) 1913-1927. DOI: 10.1016/S0013-7944(03)00131-0.
- [55] Gómez, F. J., Elices, M., A fracture criterion for sharp V-notched samples, *Int. J. Fracture*, 123 (2003) 163-175. DOI: 10.1023/B:FRAC.0000007374.80996.a2.
- [56] Gómez, F.J., Elices, M., A fracture criterion for blunted V-notched samples, *Int. J. Fracture*, 127 (2004) 239-264. DOI: 10.1023/B:FRAC.0000036832.29429.21.
- [57] Gómez, F.J., Elices, M., Planas, J., The cohesive crack concept: application to PMMA at  $-60^{\circ}\text{C}$ , *Eng. Fract. Mech.*, 72 (2005)1268-1285. DOI: 10.1016/j.engfracmech.2004.09.005.
- [58] Gómez, F.J., Elices, M., Berto, F., Lazzarin, P., Local strain energy to asses the static failure of U-notches in plates under mixed mode loading. *Int. J. Fracture*, 145 (2007) 29-45. DOI: 10.1007/s10704-007-9104-3.
- [59] Berto, F., Lazzarin, P., Gomez, F.J., Elices, M., Fracture assessment of U-notches under mixed mode loading: Two procedures based on the 'equivalent local mode I' concept, *Int. J. Fracture*, 148 (2007) 415-433. DOI: 10.1007/s10704-008-9213-7.
- [60] Lazzarin, P., Berto, F., Elices, M., Gómez, J., Brittle failures from U- and V-notches in mode I and mixed, I+II, mode. A synthesis based on the strain energy density averaged on finite size volumes, *Fatigue Fract. Eng. Mater. Struct.*, 32 (2009) 671-684. DOI: 10.1111/j.1460-2695.2009.01373.x.
- [61] Gómez, F.J., Elices, M., Berto, F., Lazzarin, P., Fracture of V-notched specimens under mixed mode (I+II) loading in brittle materials, *Int. J. Fracture*, 159 (2009), 121-135. DOI: 10.1007/s10704-007-9104-3.
- [62] Erdogan, F., Sih, C.G., On the crack extension in plates under plane loading and transverse shear. *J. Basic Eng.*, (1963) 519-525. DOI: 10.1115/1.3656897.
- [63] Schleicher, F., Der Spannungszustand an der Fließgrenze (Plastizitätsbedingung), *Zeitschrift für angewandte Mathematik und Mechanik*, 6 (3) (1926) 199–216.
- [64] Berto, F., Lazzarin, P., Recent developments in brittle and quasi-brittle failure assessment of engineering materials by means of local approaches, 75 (2014) 1-48. DOI: 10.1016/j.msar.2013.11.001.
- [65] Filippi, S., Lazzarin, P. Tovo, R., Developments of some explicit formulas useful to describe elastic stress fields ahead of notches in plates, *Int. J. Solids Struct.*, 39 (2002) 4543-4565. DOI: 10.1016/S0020-7683(02)00342-6.
- [66] Williams, M.L., Stress singularities resulting from various boundary conditions in angular corners on plates in tension, *J. Appl. Mech.*, 19 (1952), 526-528.
- [67] Creager, M., Paris, P.C., Elastic field equations for blunt cracks with reference to stress corrosion cracking, *Int. J. Fract. Mech.*, 3 (1967) 247–252. DOI: 10.1007/BF00182890.
- [68] Gross, R., Mendelson, A., Plane Elastostatic Analysis of V-Notched Plates, *Int. J. Fract. Mech.*, 8 (1972) 267-27. DOI 10.1007/BF00186126.
- [69] Lazzarin, P., Filippi, S., A generalised stress intensity factor to be applied to rounded V-shaped notches, *Int. J. Solids Struct.*, 43 (2006) 2461-2478. DOI: 10.1016/j.ijsolstr.2005.03.007.
- [70] Lazzarin, P., Berto, F., Gómez, F.J., Zappalorto, M., Some advantages derived from the use of the strain energy density over a control volume in fatigue strength assessments of welded joints, *Int. J. Fatigue*, 30 (2008) 1345-1357. DOI: 10.1016/j.ijfatigue.2007.10.012.
- [71] Lazzarin, P., Berto, F. Zappalorto, M., Mean values of the Strain Energy Density from coarse meshes: theoretical bases of a method suitable for rapid calculations of notch stress intensity factors and fatigue strength of welded joints, *Int. J. Fatigue*, 32 (2010) 1559-1567. DOI: 10.1016/j.ijfatigue.2010.02.017.
- [72] Lazzarin, P., Berto, F., Radaj, D. Stress intensity factors of welded lap joints: a comparison between pointed slit based and keyhole notch based models, with extensions to strain energy density and J-integral, *Fatigue Fract. Eng. Mater. Struct.*, 32 (2009) 713-735. DOI: 10.1111/j.1460-2695.2009.01379.x.
- [73] Kotousov, A., Wang, C.H. Three dimensional stress constraint in an elastic plate with a notch, *Int. J. Solids Struct.*, 39 (2002) 4311-4326. DOI: 10.1016/S0020-7683(02)00340-2.
- [74] Berto, F., Lazzarin, P., Wang, C.H. Three-dimensional linear elastic distributions of stress and strain energy density ahead of V-shaped notches in plates of arbitrary thickness, *Int. J. Fracture*, 127 (2004) 265-282. DOI: 10.1023/B:FRAC.0000036846.23180.4d.
- [75] Kotousov, A., Lew, T.L., Stress singularities resulting from various boundary conditions in angular corners of plates of arbitrary thickness in extension, *Int. J. Solids Struct.*, 43(2005) 5100-5109. DOI: 10.1016/j.ijsolstr.2005.06.037.
- [76] Kotousov, A., Fracture in plates of finite thickness, *Int. J. Solids Struct.*, 44 (2007) 8259-8273. DOI: 10.1016/j.ijsolstr.2007.06.011.



- [77] Harding, S., Kotousov, A., Lazzarin, P., Berto, F., Transverse Singular Effects in V-shaped Notches Stressed in Mode II, *Int. J. Fracture*, 164 (2009) 1-14. DOI: 10.1007/s10704-010-9449-x.
- [78] B. Atzori, V. Dattoma, A comparison of the fatigue behaviour of welded joints in steels and in aluminium alloys. IIW Doc XXXIII, (1983) 1089-1983.
- [79] Radaj, D., *Design and Analysis of Fatigue Resistant Welded Structures*. Abington Publishing, Cambridge (1990). ISBN-9781855730045.