



Models of initiation fatigue crack paths proposed by Macha

Z. Marciniak

Opole University of Technology, Poland
z.marciniak@po.opole.pl

D. Rozumek

Opole University of Technology, Poland
d.rozumek@po.opole.pl

ABSTRACT. Professor E. Macha devoted his academic life to solving the problems connected with random multiaxial fatigue in components of machines and structures. In his studies he formulated stress, strain and energy criteria related to critical plane concept. He also proposed several methods to determine critical plane position. In particular, he formulated and verified weight functions applied in order to determine critical plane position. The variance method constituted another significant contribution to the development of methods for determining critical plane position. Apart from these criteria, Macha was exploring energy approach in fatigue of materials and the development of fatigue cracks.

He has also observed that strain characteristics multiplied by stress amplitude determined at specimen half-life are applied to estimate fatigue life using energy criteria. However, for cyclically instable materials, stress amplitude value may differ a lot; therefore he proposed the method to determine energy fatigue characteristics directly from experimental research.

KEYWORDS. Multiaxial fatigue criteria; Energy; Crack growth; Variance method; Weight function.

INTRODUCTION

Predicting service life of different objects is a very important issue for modern engineering. Wrong service life estimation may result in accidents and disasters. Therefore, studies aimed to understand and control this phenomenon, started already in the 19th century, are continued today. The multitude of problems connected with it suggests that scientists still have plenty of work ahead. Initially, the scope of studies was limited to uniaxial, constant-amplitude issues only. With increasing knowledge on the phenomenon, the interest in multiaxial fatigue (most frequent in engineering practice) was growing. At the same time, many stress assessment criteria were proposed. Another step in the development involved attempts to assess life for random loads. This problem was explored by Professor Macha as well [1-4]. He started his work from proposing mathematical models to assess fatigue life for materials in the conditions of random complex stress state, where besides stress criteria he demonstrated the method for determining critical plane position using weight functions. Further studies were connected with strain and energy criteria, and methods used to



determine critical plane position. Many scientists became interested in these studies, which resulted in numerous contacts and team work, e.g. with Carpinteri [5, 6], Sakane [7], Sonsino [8], Dragon [9], Petit [9], and others. The purpose of this study is to present academic achievements of Professor Macha.

MATHEMATICAL MODELS AND THEIR EXPERIMENTAL VERIFICATION

One of the first criteria proposed by E. Macha for multiaxial random loads [1, 2] has the following form:

$$\max_t [B\tau_{ns}(t) + K\sigma_n(t)] = F \quad (1)$$

where $\tau_{ns}(t)$ and $\sigma_n(t)$ are: shear stress and normal stress in fracture plane, respectively; and B, K, F – constants for the selection of a given criterion version. Initially, in this criterion fracture plane was regarded as the critical plane. However subsequent analyses make it possible to observe that this plane changes especially for elastic-plastic materials.

Detailed criterion guidelines are: (i) fatigue crack is generated (caused) by the activity of normal stresses $\sigma_n(t)$ and shear stresses $\tau_{ns}(t)$ in the direction \vec{s} in plane with normal \vec{n} , (ii) direction \vec{s} is concurrent with average direction of shear stresses.

In the criteria related to critical plane it is very important to determine critical plane position. In order to determine its position, it was proposed to apply the weight function method.

The weight function method involves finding averaged positions of main axes directions through properly selected weight functions W_k .

$$\hat{l}_1 = \cos \frac{1}{W} \sum_{k=1}^L \alpha_{1k} W_k, \quad \hat{m}_2 = \cos \frac{1}{W} \sum_{k=1}^L \beta_{2k} W_k, \quad \hat{n}_3 = \cos \frac{1}{W} \sum_{k=1}^L \gamma_{3k} W_k, \quad (2)$$

where:

$$W = \sum_{k=1}^L W_k - \text{sum of weights,}$$

L – number of averages,

$\alpha_1, \beta_2, \gamma_3$ – angles between main stresses and axes in the Cartesian coordinates, (σ_1, x) , (σ_2, y) , (σ_3, z) , respectively

Then, critical plane position is being determined relative to these averaged directions.

6 weight functions are demonstrated in the study [1]:

- Weight I – $W_k = 1$ – it is assumed that each position of the main axes has the same effect on the critical plane position,

- Weight II – $W_k = \frac{\sigma_{1k} - \sigma_{1\min}}{\sigma_{1\max} - \sigma_{1\min}}$ for $k = 1, 2, \dots, N$ – this weight reduces the impact of maximum main stress $\sigma_1(t)$ value on the critical plane position,

- Weight III – $W_k = \begin{cases} 0 & \text{for } \sigma_{1k} < a \cdot \sigma_{af} \\ 1 & \text{for } \sigma_{1k} \geq a \cdot \sigma_{af} \end{cases}$ $0 < a \leq 1$ for $k = 1, 2, \dots, N$ – according to this weight, only those

positions of main axes are averaged, for which maximum stress value is $\sigma_1(t) \geq a \cdot \sigma_f$, where σ_f is fatigue limit,

- Weight IV – $W_k = \begin{cases} 0 & \text{for } \sigma_{1k} < \nu \cdot R_e \\ 1 & \text{for } \sigma_{1k} \geq \nu \cdot R_e \end{cases}$ for $k = 1, 2, \dots, N$ – only those positions of main axes are averaged,

for which maximum stress value $\sigma_1(t)$ is higher than product of Poisson's ratio and yield point,



$$\text{- Weight V - } W_k = \begin{cases} \frac{\sigma_{1k} - a\sigma_f}{\sigma_{1k\max} - \sigma_{1\min}} & \text{for } \sigma_{1k} \geq a \cdot \sigma_{af} \\ 0 & \text{for } \sigma_{1k} < a \cdot \sigma_{af} \end{cases} \quad \text{-- this weight was developed as a result of combining weights}$$

II and III,

$$\text{- Weight VI - } W_k = \begin{cases} \left(\frac{\sigma_{1k}}{a\sigma_f} \right)^m & \text{for } \sigma_{1k} \geq a \cdot \sigma_{af} \\ 0 & \text{for } \sigma_{1k} < a \cdot \sigma_{af} \end{cases} \quad \text{-- in this proposal only those positions of main axes are taken for}$$

averaging, in which $\sigma_1(t)$ is higher than fatigue limit fraction, while their share is in exponential function dependent on Wöhler curve inclination.

However, the selection of proper angles for averaging creates problems, and there are no physical guidelines, which angles should be averaged. The issue of averaging proper angles is discussed in the study [10], where direction cosines were made dependent on Euler angles. Matrix of direction cosines defined in this way is expressed in the following form

$$\begin{bmatrix} \cos \varphi \cos \theta \cos \psi - \sin \varphi \sin \psi & -\cos \varphi \cos \theta \sin \psi - \sin \varphi \cos \psi & \cos \varphi \sin \theta \\ \sin \varphi \cos \theta \cos \psi + \cos \varphi \sin \psi & -\sin \varphi \cos \theta \sin \psi + \cos \varphi \cos \psi & \sin \varphi \sin \theta \\ -\sin \theta \cos \psi & \sin \theta \sin \psi & \cos \theta \end{bmatrix}. \quad (3)$$

Nevertheless, some transformations are required in order to obtain values of Euler angles. The first step involves calculation of the quantity:

$$\chi = \arccos \frac{1}{2}(l_1 + m_2 + n_3 - 1), \quad u_1 = \frac{m_3 - n_2}{2 \sin \chi}, \quad u_2 = \frac{n_1 - l_3}{2 \sin \chi}, \quad u_3 = \frac{l_2 - m_1}{2 \sin \chi}. \quad (4)$$

Then, Euler - Rodriguez parameters are used:

$$\lambda = u_1 \sin \frac{\chi}{2}, \quad \mu = u_2 \sin \frac{\chi}{2}, \quad \nu = u_3 \sin \frac{\chi}{2}, \quad \rho = \cos \frac{\chi}{2} \quad (5)$$

to determine values of angles

$$\varphi = \operatorname{arctg} \left(\frac{\nu}{\rho} \right) - \operatorname{arctg} \left(\frac{\lambda}{\mu} \right), \quad \theta = \arcsin \left(\frac{m_3}{\sin \varphi} \right), \quad \psi = \operatorname{arctg} \left(\frac{\nu}{\rho} \right) + \operatorname{arctg} \left(\frac{\lambda}{\mu} \right). \quad (6)$$

Euler angles calculated in this way are averaged using the following relations:

$$\hat{\varphi} = \frac{1}{W} \sum_{k=1}^L \varphi(k) W(k), \quad \hat{\theta} = \frac{1}{W} \sum_{k=1}^L \theta(k) W(k), \quad \hat{\psi} = \frac{1}{W} \sum_{k=1}^L \psi(k) W(k). \quad (7)$$

Then, Macha and Będkowski [11] developed variance method to determine critical plane position. In this method, the critical plane is considered to be the plane, for which the variance of equivalent stress reduced by selected criterion reaches maximum.

The study [12] contains comparison of lives of steel specimens using variance method with damage accumulation method for criterion of maximum shear stress in the critical plane. According to this criterion, the equivalent stress $\sigma_{eq}(t)$ takes the following form

$$\sigma_{eq}(t) = \sigma_x(t) \cdot \sin(2\alpha) - 2\tau_{xy}(t) \cdot \cos(2\alpha), \quad (8)$$

where: $\sigma_x(t)$ - normal stress along the specimen axis,
 $\tau_{xy}(t)$ - shear stress in the specimen cross section,
 α - angle determining the critical plane position.

From Eq. (8) it appears that the equivalent stress $\sigma_{eq}(t)$ is linearly dependent on the stress state components $\sigma_x(t)$ and $\tau_{xy}(t)$, so it can be expressed as

$$\sigma_{eq} = \sum_{j=1}^n a_j x_j = a_1 x_1 + a_2 x_2, \quad (9)$$

where: $a_1 = \sin(2\alpha)$, $a_2 = 2\cos(2\alpha)$, $x_1 = \sigma_x$, $x_2 = \tau_{xy}$.

From theory of probability [13] it results that the variance of random variable being a linear function of some random variables is expressed by the following formula

$$\mu_{\sigma_{eq}} = \sum_{j=1}^n a_j^2 \mu_{x_j} + 2 \sum_{j \neq k} a_j a_k \mu_{x_j x_k} = a_1^2 \mu_{x_1} + a_2^2 \mu_{x_2} + 2a_1 a_2 \mu_{x_1 x_2}, \quad (10)$$

where: $\mu_{\sigma_{eq}}$ - variance of equivalent stress σ_{eq} ,
 μ_{x_1} - variance of normal stress σ_x ,
 μ_{x_2} - variance of shear stress τ_{xy} ,
 $\mu_{x_1 x_2}$ - covariance of normal σ_x and shear stress τ_{xy} stresses.

Under biaxial random stationary and ergodic stress state, the variances μ_{x_1} , μ_{x_2} and the covariance $\mu_{x_1 x_2}$ in Eq. (10) are constant.

In the method of variance for determination of the critical plane position the maximum function of Eq. (10) is searched in relation of the angle α occurring in coefficients a_1 and a_2 . After reduction, the variance of equivalent stress $\mu_{\sigma_{eq}}$ versus the angle α can be written as

$$\mu_{\sigma_{eq}} = \sin^2(2\alpha) \mu_{x_1} + 4 \cos^2(2\alpha) \mu_{x_2} + 2 \sin(4\alpha) \mu_{x_1 x_2}. \quad (11)$$

An exemplary assessment of the critical plane position for loading combination K01 [12] obtained using the variance and damage accumulation methods is shown in Fig. 1.

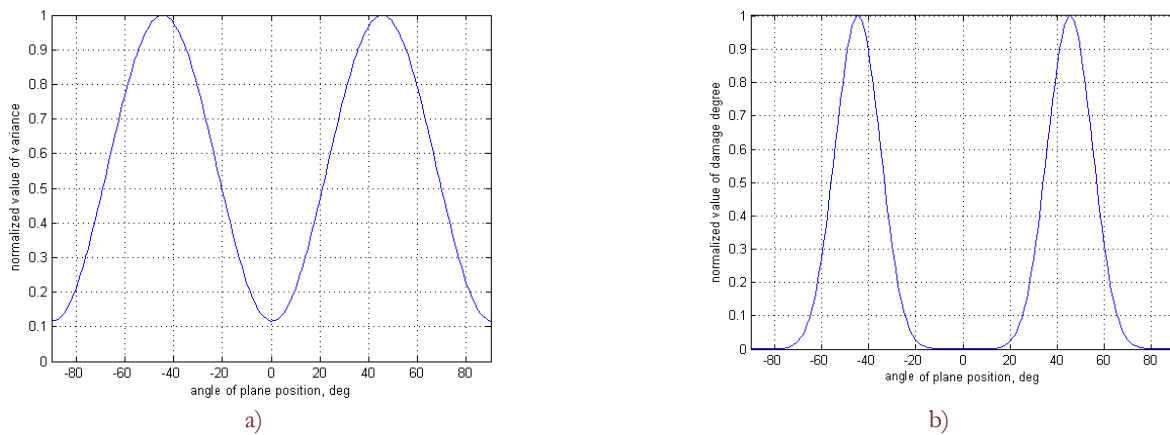


Figure 1: Dependence of the normalized value of: a) variance, b) damage accumulation on the angle α of critical plane position for loading combination K01 ($\lambda_\sigma = 0.189$) [14].



Fig. 2 compares calculated life values and experimental values for variance and damage accumulation [12].

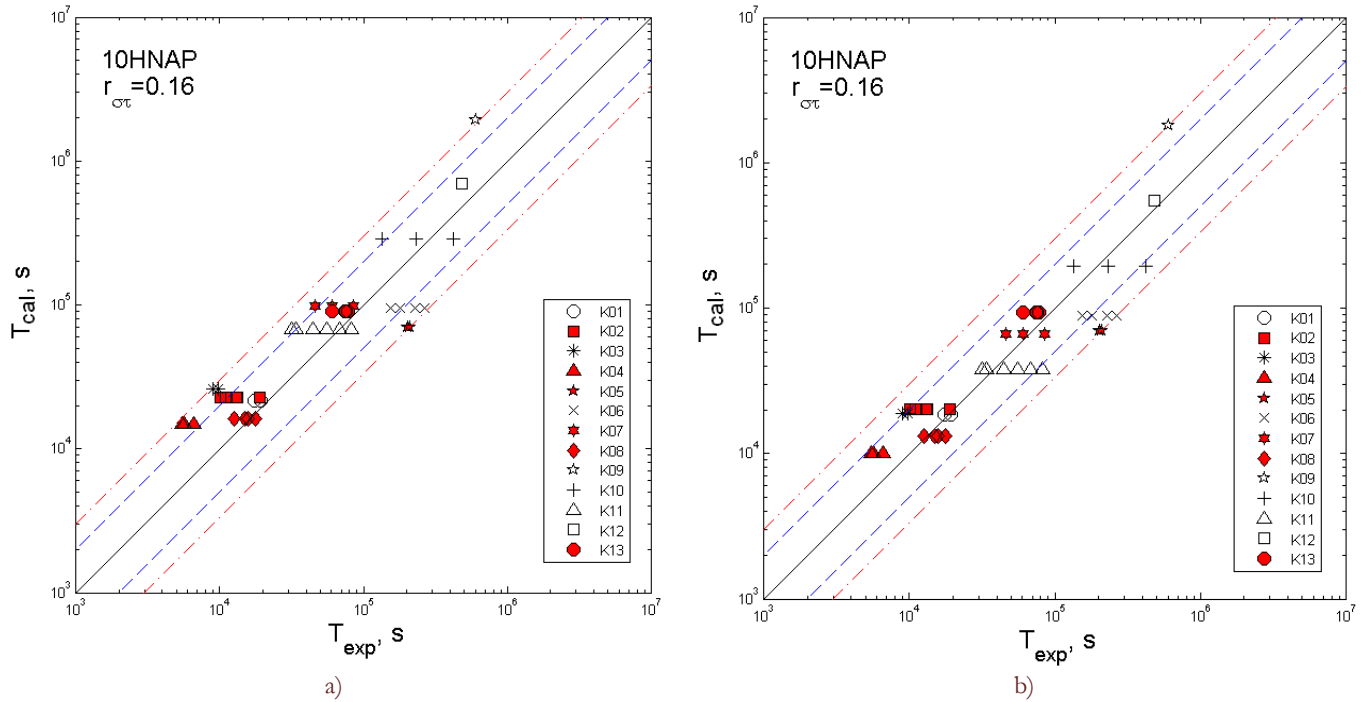


Figure 2: Comparison of calculated and experimental fatigue lives with critical planes determined according to: a) variance and b) damage accumulation methods [14].

Strain fatigue criterion [3] expressed as

$$\max_t [b\varepsilon_m(t) + k\varepsilon_n(t)] = q \quad (12)$$

is another proposal to formulate multiaxial random fatigue in the field of strains, where $\varepsilon_{ns}(t)$ and $\varepsilon_n(t)$ are shear and normal strain in critical plane, respectively; and a, b, k, q – constants for the selection of a given criterion version.

In 1991, Macha, Grzelak and Łagoda [14] attempted to apply spectral method to determine fatigue life. Studies on these issues were continued further in cooperation with Nieslony [15]. In these studies, assuming linear effort criteria, a generalised spectral method was formulated for determining fatigue life of materials put to multiaxial loading, using the function of power spectral density in the field of frequency. Multiaxial state of stress is reduced to uniaxial state, and accumulation of damage is carried out using standard material characteristics. The study proves that the results for lives assessed using spectral method in the field of frequency and cycle counting method in the field of time are much the same. Whereas, determination of expected critical plane position using variance method for time histories gives results equivalent to the function of power spectral density.

Then, Professor Macha focused his attention on stress distribution in notch root. Like in Neuber [16] and Molski-Glinka [17] criteria, Łagoda-Macha [18] proposed an energy equation for determining the state of stresses in notch bottom as

$$W_{LM} = \frac{\sigma_{\max}^2}{2E} + \frac{1-n'}{1+n'} \sigma_{\max} \left(\frac{\sigma_{\max}}{K'} \right)^{\frac{1}{n'}}, \quad (13)$$

where: n' - exponent of cyclic strain curve, K' - coefficient of cyclic strain curve.

Experimental verification proved that the values obtained through this relation are between the results obtained using Neuber and Molski-Glinka relations.

Tab. 1 contains sample calculation results for the above three models [19].



K_t	Model		
	Neuber σ_{\max} (MPa)	Molski-Glinka σ_{\max} (MPa)	Macha-Łagoda σ_{\max} (MPa)
9.61	421	386	405
4.30	329	302	316
3.23	294	270	282
1.85	233	215	224

Table 1: The presentation σ_{\max} depending on K_t and strain energy density models.

Professor Macha was interested most in energy criteria of multiaxial random fatigue. In this field, in cooperation with Łagoda he proposed a generalised criterion of energy density parameter for normal and shear strains in critical plane, shown as [18, 20]

$$\max_t [\beta W_{ns}(t) + \kappa W_n(t)] = Q \quad \text{or} \quad \max_t [W_{eq}(t)] = Q \quad (14)$$

where β , κ , Q – constants for the selection of a given criterion version.

Guidelines of the proposed criterion are as follows [21]:

- “a) this portion of strain energy density is responsible for fatigue crack, which matches the work of normal stress $\sigma_n(t)$ in normal strain $\varepsilon_n(t)$, that is $W_n(t)$ and work of shear stress $\tau_{ns}(t)$ in a shear strain $\varepsilon_{ns}(t)$ in the direction s in plane with normal n , that is $W_{ns}(t)$,
- b) direction s in the critical plane matches average direction, in which density of shear strain energy is maximal,
- c) in boundary state, material effort is determined by the maximum value of linear combination of energy parameters $W_n(t)$ and $W_{ns}(t)$.”

For uniaxial stress state, strain energy density parameter is expressed as

$$W(t) = 0.5\sigma(t)\varepsilon(t)\text{sgn}[\sigma(t), \varepsilon(t)] = 0.5\sigma(t)\varepsilon(t) \frac{\text{sgn}[\sigma(t)] + \text{sgn}[\varepsilon(t)]}{2} \quad (15)$$

For multiaxial stress state, the course of equivalent strain energy density parameter is calculated in the critical plane with normal n and shear direction s as

$$W_{eq}(t) = \beta W_{ns}(t) + \kappa W_n(t) = 0.5\kappa\sigma_n(t)\varepsilon_n(t)\text{sgn}[\sigma_n(t), \varepsilon_n(t)] + 0.5\beta\tau_{ns}(t)\varepsilon_{ns}(t)\text{sgn}[\tau_{ns}(t), \varepsilon_{ns}(t)] \quad (16)$$

The proposed energy criterion in the critical plane is applicable for cyclic and random loads for small and large number of cycles. Depending on the coefficients chosen, different criteria are obtained and thus, for:

- $\beta = 0$, $\kappa = 1$ we have the criterion of maximum energy density for normal strain in the critical plane,
- $\beta = 1$, $\kappa = 0$ we have the criterion of maximum energy density for shear strain in the critical plane,
- $\beta = 1$, $\kappa = 1$ we have the criterion of maximum energy density for normal and shear strain in the critical plane.

When applying energy fatigue criteria to assess life, energy characteristics are used, developed as a result of the Coffin-Manson-Basquin characteristic multiplication [22-24] by stress amplitude determined for specimen half-life. However, this characteristic not fully illustrates the behaviour of cyclically unstable materials. Being aware of these differences, Professor Macha and Slowik proposed a new model to determine energy fatigue characteristics directly from experimental research. This model is described as [25]

$$W(t) = 0.5\sigma(t) \cdot \left| \varepsilon(t) - \varepsilon_i^{pl} \right|, \quad (17)$$

where $\varepsilon_i^{pl} = \varepsilon(t_i)$ for $\sigma(t_i) = 0$ and $i = 1, 2, 3, \dots$ are successive numbers of the hysteresis loop (σ - ε).



In Eq. (17), $W(t)$, $\sigma(t)$, $\varepsilon(t)$ are continuous functions of time t , and ε_i^{pl} and ε_{i+1}^{pl} are constant values in time t in the hysteresis loop with the number i , while ε_i^{pl} is the plastic strain registered in the moment t_i , when the stress $\sigma(t_i)$ is equal to zero, and remains constant to the moment t_{i+1} when the stress reaches zero again, i.e. $\sigma(t_{i+1}) = 0$. Then the new registered value of plastic strain ε_{i+1}^{pl} replaces the previous one ε_i^{pl} . This procedure is repeated for each cycle of loading.

Fig. 3 shows sample hysteresis loops and energy parameter course calculated on the basis of variable-amplitude history of stresses and strains.

Energy parameter course calculation procedure for variable-amplitude loads:

Step 1. In point 0, individual values of stresses, strains and energy parameter are: $\sigma(t_0) = 0$, $\varepsilon(t_0) = 0$, $\varepsilon_0^{pl} = 0$, $W(t_0) = 0$.

Step 2. In point A, individual parameters have the following values: $\sigma(t_A) = \sigma_A$, $\varepsilon(t_A) = \varepsilon_A$, $\varepsilon_A^{pl} = \varepsilon_0^{pl} = 0$, $W(t_A) = 0.5 \cdot \sigma_A \cdot |\varepsilon_A - \varepsilon_0^{pl}| = 0.5 \cdot \sigma_A \cdot \varepsilon_A$.

Step 3. Then, going to point B we obtain: $\sigma(t_B) = \sigma_B = 0$, $\varepsilon(t_B) = \varepsilon_B = 0$, $\varepsilon_B^{pl} = \varepsilon_B^{pl}$, $W(t_B) = 0.5 \cdot \sigma_B \cdot |\varepsilon_B - \varepsilon_B^{pl}| = 0$.

Step 4. Point C: $\sigma(t_C) = \sigma_C$, $\varepsilon(t_C) = \varepsilon_C$, $\varepsilon_C^{pl} = \varepsilon_B^{pl}$, $W(t_C) = 0.5 \cdot \sigma_C \cdot |\varepsilon_C - \varepsilon_B^{pl}|$.

Step 5. Point D: $\sigma(t_D) = \sigma_D = 0$, $\varepsilon(t_D) = \varepsilon_D = 0$, $\varepsilon_D^{pl} = \varepsilon_D^{pl}$, $W(t_D) = 0.5 \cdot \sigma_D \cdot |\varepsilon_D - \varepsilon_D^{pl}| = 0$.

Step 6. Point E: $\sigma(t_E) = \sigma_E$, $\varepsilon(t_E) = \varepsilon_E$, $\varepsilon_E^{pl} = \varepsilon_D^{pl}$, $W(t_E) = 0.5 \cdot \sigma_E \cdot |\varepsilon_E - \varepsilon_D^{pl}|$, etc.

Fig. 3d presents energy parameter course calculated based on above procedure.

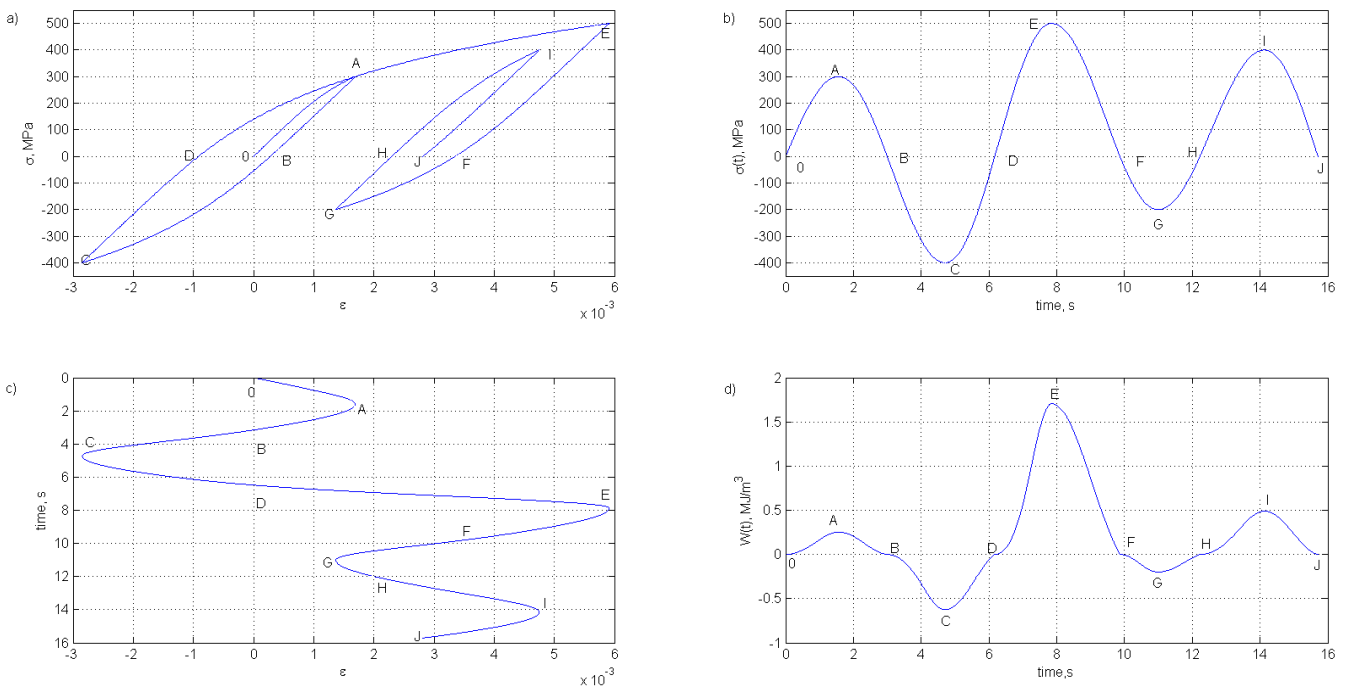


Figure 3: Sample hysteresis loops a), stress courses b), strain courses c), energy parameter courses d).

Fig. 4 shows energy fatigue characteristic for steel C45 according to the formula (17).

The models and methods proposed above were used to assess fatigue life until crack initiation. Whereas, as regards development of fatigue cracks, Rozumek and Macha proposed an energy criterion based on parameter J for three crack modes [26]. This criterion was verified for mode I and mode III [27].

$$\left(\frac{\Delta J_I}{J_{Ic}}\right)^2 + \left(\frac{\Delta J_{II}}{J_{IIc}}\right)^2 + \left(\frac{\Delta J_{III}}{J_{IIIc}}\right)^2 = 1, \quad (18)$$

where J_{Ic} , J_{IIc} , J_{IIIc} are critical values for modes I, II and III.

The criterion (18) was successfully verified while tests of aluminium alloy and steels.

Different bending (cracking mode I) to torsion (cracking mode III) ratio in steel 18G2A is shown in Fig. 5 [28]. It provides grounds to observe shift of experimental points towards increasing the value of parameter ΔJ_I , except of the

angle $\alpha = 60^\circ$, where decrease in these values was confirmed. Considerable increase of parameter ΔJ_I and ΔJ_{III} values was observed with rising fatigue crack growth rate (Fig. 5, curves 1 to 2).

Diagrams 1 and 2 shown in Fig. 5 concern fatigue crack growth rates: $da/dN = 1.68 \cdot 10^{-8}$ m/cycle and $da/dN = 4.23 \cdot 10^{-8}$ m/cycle, respectively.

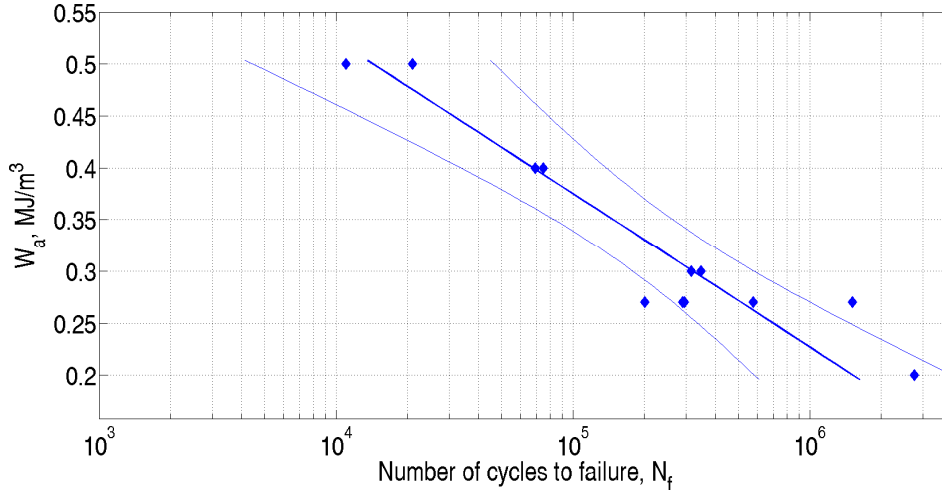


Figure 4: Energy fatigue characteristic for steel C45.

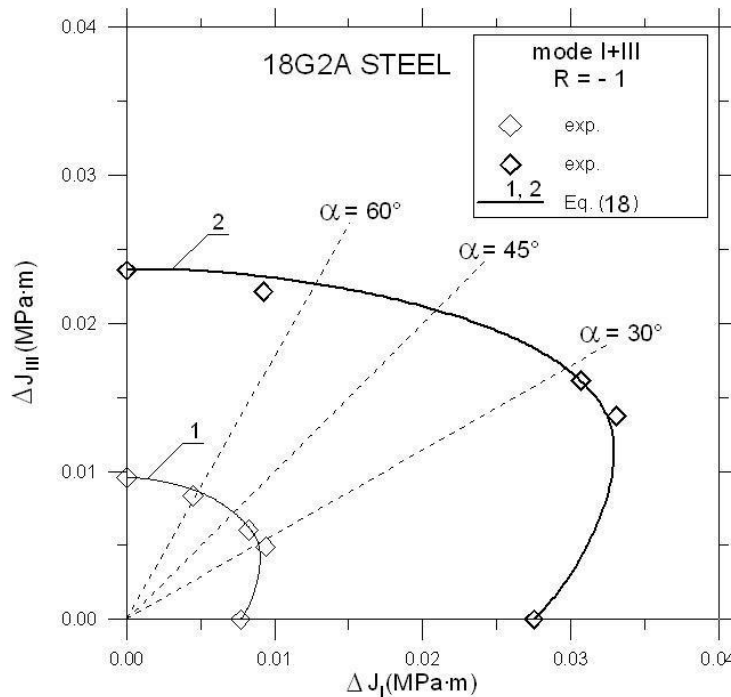


Figure 5: Comparison of experimental results for different bending to torsion ratios with those calculated according to the Eq. (18) for 18G2A steel [27].

Different bending (cracking mode I) to torsion (cracking mode III) ratio in AlCuMg1 alloy is shown in Fig. 6.

Fig. 6 [28] provides grounds to observe shift of experimental points towards increasing the values of parameter ΔJ_I - these increment values were lower than for steel 18G2A. Experimental results of interdependences between cracking mode I and III, for constant da/dN ratio value were defined by Eq. (18). Diagrams 1 and 2 shown in Fig. 6 concern fatigue crack growth rates: $da/dN = 7.64 \cdot 10^{-8}$ m/cycle and $da/dN = 1.41 \cdot 10^{-7}$ m/cycle, respectively.

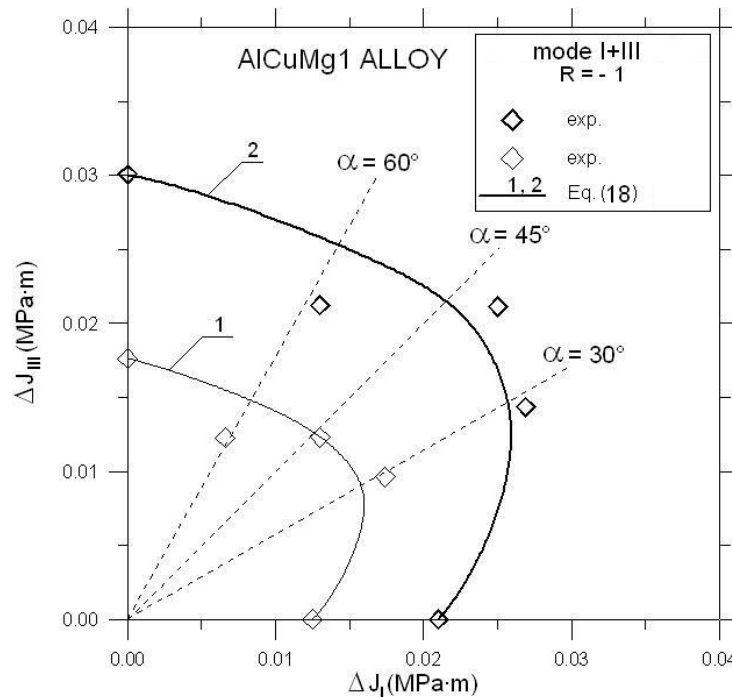


Figure 6: Comparison of experimental results for different bending to torsion ratios with those calculated according to the Eq. (18) for AlCuMg1 alloy.

SUMMARY

This brief description of professor Macha activity irrefutably proves his wide interests and great influence on progress regarding the issues of fatigue life assessment for components of machines and structures. During his academic career, Macha with colleagues proposed many fatigue criteria concerning the parameters of stress, strain and strain energy density both in the field of time and frequency. Macha's interests covered initiation range and propagation of fatigue cracks. Many times these criteria were verified in various load conditions for different materials, and were presented during various scientific conferences.

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