Interpreting experimental fracture toughness results of quasi-brittle natural materials through multi-parameter approaches

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ABSTRACT. Natural stones like marbles are often employed as façade panels to externally cover buildings. These natural materials tend to exhibit a quasi-brittle nonlinear fracture behaviour which, conversely to concrete counterpart, has much less been studied in the literature. In the present paper, an experimental campaign on the so-called red Verona marble is carried out, and the results are discussed together with some previously published results on the white Carrara marble. The analysis of the two marbles as a whole allows us to discuss size effect and to point out the need for additional parameters in order to describe their fracture behaviour. The study focuses on a two-parameter model which accounts for a characteristic material length. A quantitative correlation between material microstructure features, obtained from thin sections technique, and the characteristic material length is proposed.

KEYWORDS. Fracture process zone; Marble; Quasi-brittle behaviour; Size effect; Two-parameter model.

INTRODUCTION

Quasi-brittle materials are typically characterized by a stable nonlinear load-deflection response followed by an unstable fracture. Such a behaviour is generally attributed to the development of a finite-size process zone ahead of the crack tip. The quasi-brittle behaviour has largely been examined for concrete both experimentally and theoretically [1].

When fracture toughness in quasi-brittle materials has to be determined, an effective crack length (spanning a traction-free segment and a stressed segment embedded in the process zone) is needed to be defined, with respect to which the critical stress intensity factor has to be computed. Accordingly, in the realm of concrete material testing, Shah and co-workers [2] proposed a two-parameter method to calculate the effective crack length in the verge of unstable cracking. Other theories are available in the literature to describe experimental size-dependent fracture data. These theories (e.g. theory of critical distances [3,4], finite fracture mechanics [5], cohesive crack models [6], size effect laws [7-10], non-local stress failure criteria [11], and crack extension resistance curves [12]) are characterized by a fil rouge, that is to say, the near-crack tip stress field and/or the fracture toughness is described by means of multi-parameter approaches. A size-dependence discussion of the RILEM work-of-fracture method can be found in Ref. [13].

Natural materials, like calcite marbles used for building claddings [14, 15], often exhibit a quasi-brittle mechanical behaviour [16, 17] which, conversely to concrete counterparts, has much less been investigated despite the fact that, under different deteriorating in-service actions (temperature cycles, wind and seismic loads, chemical actions, humidity changes, etc.), fracture behaviour might significantly affect structural failure.

In the present paper, an experimental campaign on the so-called red Verona marble is performed, and the results are examined with previously published results on the white Carrara marble [18]. The analysis of the two marbles allows us to discuss size effect and to point out the need for additional parameters in order to describe their fracture behaviour. The study focuses on a two-parameter model which accounts for a characteristic material length. A tentative quantitative correlation between material microstructure features, obtained from thin sections technique, and the characteristic material length is presented.

MATERIAL MICROSTRUCTURAL CHARACTERIZATION AND EXPERIMENTAL MECHANICAL TESTS

The results of some fracture experimental tests on two different types of natural stones, used as cladding elements in building façades, are presented. In particular, white Carrara marble, which is extracted from Carrara area in the Alpi Apuane metamorphic complex, and red Verona marble, which is a variety of limestone sedimentary rock taking its name from the city of Verona in Northern Italy, are analysed. For the former material, the results published in Refs [18,19] are herein elaborated, while the experimental results for the latter are presented in this paper for the first time. A picture of a thin section of Carrara marble is shown in Fig. 1. From a qualitative viewpoint, the picture indicates common features of calcite grains with granoblastic texture and anhedral grain shape. The grain size distribution is characterized by the prevalence of large grains (about 200 μm on average), followed by smaller grains (30–50 μm). Grain boundaries are typically ranging from polygonal shape to interlobate one in both large and small crystals. A third group of calcite grains is detected as fine grained aggregates (<10 μm) commonly rimming large porphyroblasts. Calcite crystals are characterized by various degree of plastic deformation, as is demonstrated by the occurrence of grains with extinction features from homogeneous to undulated; often twin lamellae occur. Twin planes or cleavage planes are well developed in most of the large grains and occur with various orientations. Two sets of twin lamellae may also occur in the same grains, testifying the activation of both the twinning planes of calcite. Locally, cleavage planes may become fracture planes never crossing through adjacent grains. No anisotropic features are present in the orthogonal sections in terms of either grain size variations, shape preferred orientation or compositional layers.

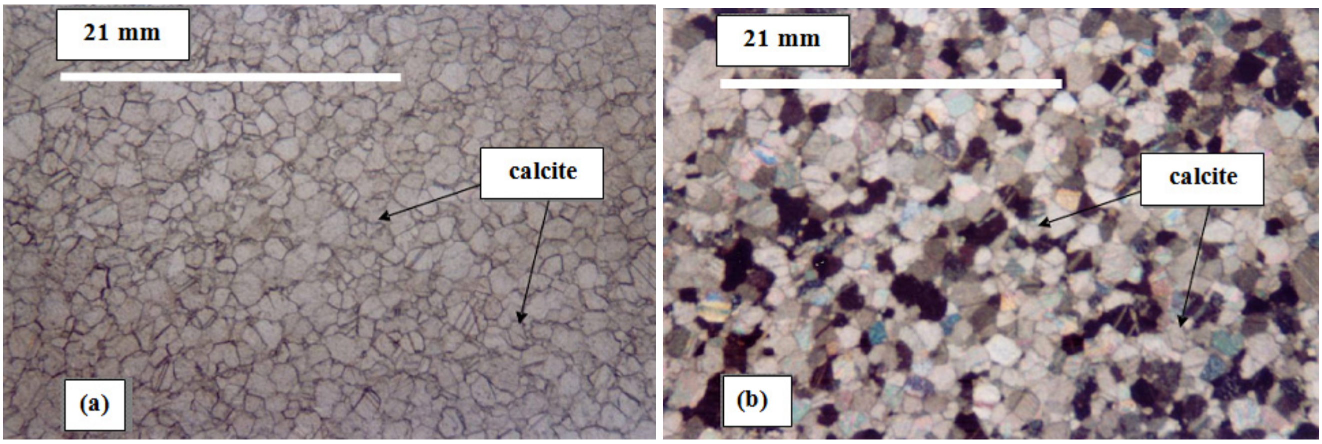

Figure 1: Carrara marble microstructure observed by a thin sections technique: (a) parallel nicol; (b) crossed nicol.

Fig. 2 shows a thin section of Verona marble. The typical structure of a sedimentar calcareous rock is evident, with pale pink nodules set in a darker pink matrix. Nodules show a wackestone texture, with skeletal grains (thin-shelled bivalves, foraminifera, gastropods, ostracodes) set in an abundant (>70%) micritic matrix (microcrystalline calcite with average size lower than 5mm). The matrix shows a packstone texture, with compacted thin-shelled bivalves (>70%) and interstitial micritic matrix. Porosity is very low (< 1%, visive estimate in thin section) with holes, variable in shape and size, empty or cemented by a mosaic of sparite (calcite crystals ranging from 10 μm to 500 μm in size).

The tensile strengths of the analysed marbles are determined from bending tests on prismatic specimens. Ten 4-point bend tests were performed on Carrara marble [18]. The nominal sizes of the specimens are 180mm in span, 60mm in depth and 30mm in width. Further, twenty 3-point bend tests are performed on Verona marble, where the nominal sizes of the specimens are 150mm in span, 30mm in depth and 60mm in width. A summary of the tensile strength is reported in Tab. 1. It can be noted that the mean value of strength is similar for the two marbles, with the Verona marble showing a comparatively larger scatter in the results. This trend can be attributed to the markedly heterogeneous nature of Verona marble, which presents weak phases of clay inclusions.

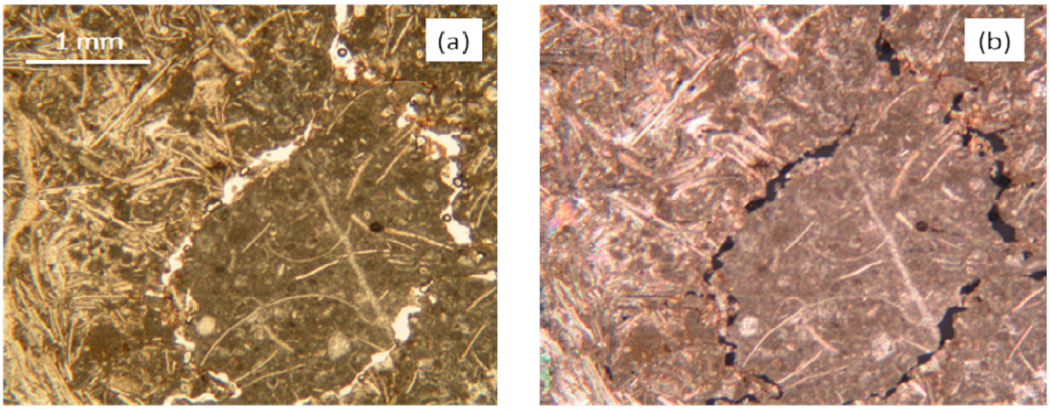

Figure 2: Verona marble microstructure observed by a thin sections technique: (a) parallel nicol; (b) crossed nicol.

	Mean value (MPa)	Stand. Dev. (MPa)	Coeff. Variation
White Carrara marble	14.91	2.97	0.20
Red Verona marble	12.27	3.55	0.29

Table 1: Tensile strength of the two marbles, obtained from bending tests

Fracture tests are then considered for the two marbles. Three-point bend tests on single edge notched beams have been performed (Fig. 3). The nominal sizes of Carrara marble specimens are [18]: depth W = 60mm, width B = W/2, span S = 3W, initial notch length a_0 = W/10. The nominal sizes of Verona marble specimens are: W = 30mm, B = W/2, S = 4W, initial notch length a_0 = W/3. The relative crack length is defined as $\alpha = a/W$.

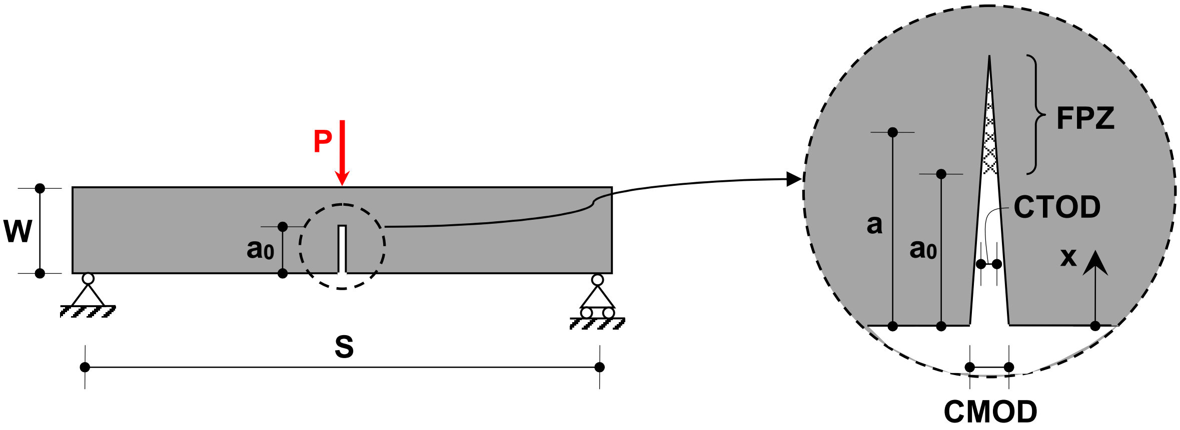

Figure 3: Three-point bend notched specimen.

During the tests, the Crack Mouth Opening Displacement (CMOD) is recorded via a clip gauge mounted across the edge notch of the beam, using two metallic gauge holders. The tests are performed through a servo-mechanical Instron 8862 machine, under a CMOD controlled procedure (CMOD rate in the range of 0.50-0.80 μm/min) in order to be able to follow the softening branch beyond the peak load. In some tests, unloading-reloading cycles in the softening phase are performed.

The experimental curves of nominal stress σ_N (equal to $6PS/(4BW^2)$) against CMOD are reported in Fig. 4 for Carrara marble and Verona marble.

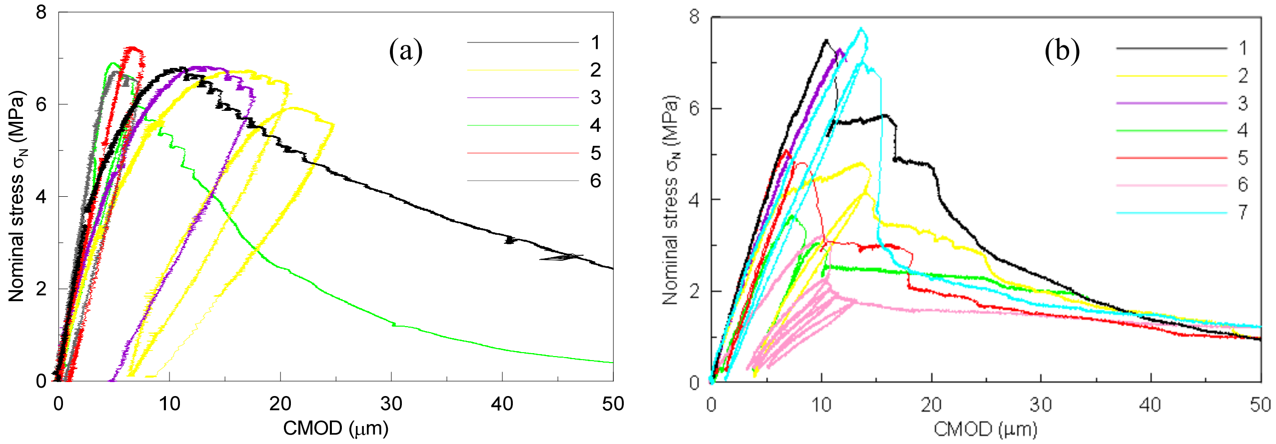

Figure 4: Experimental σ_N-CMOD curves for: (a) Carrara marble (6 tests); (b) Verona marble (7 tests).

In Tab. 2 and 3, fracture test results obtained from the elaboration of the curves plotted in Fig. 4 are illustrated. In particular, the peak nominal stress and the initial crack length are reported in the 2nd and 3rd column of both tables (the other quantities reported in Tab. 2 and 3 are described in the following Sections). Using the classical Linear Elastic Fracture Mechanics (LEFM), the nominal critical stress intensity factor K_{IC} can be worked out on the basis of the initial notch length (relative length, $\alpha = \alpha_0$). The following expression is used [20]:

$$K_I = \sigma_N \sqrt{\pi a} F(\alpha) \tag{1}$$

where

$$F(\alpha) = \frac{1}{\sqrt{\pi}} \frac{1.99 - \alpha(1-\alpha)\left(2.15 - 3.93\alpha + 2.7\alpha^2\right)}{(1+2\alpha)(1-\alpha)^{1.5}} \tag{2}$$

It can be noted that the mean values of the nominal critical stress intensity factors for the two marbles are very similar.

Specimen No.	σ_N (MPa)	a_0 (mm)	K_{IC} (MPa√m)	E (GPa)	$a-a_0$ (mm)	K_{IC}^s (MPa√m)	$CTOD_c$ (μm)	Q (mm)
1	7.06	6.07	0.98	44.6	5.35	1.32	6.9	54.98
2	6.87	6.20	0.96	28.8	5.82	1.32	11.3	60.50
3	6.95	6.40	0.99	31.1	5.29	1.31	9.9	54.86
4	7.04	6.80	1.03	50.1	3.28	1.23	4.4	32.68
5	7.38	6.57	1.06	42.6	0.88	1.12	2.4	8.33
6	6.84	6.87	1.01	85.3	3.83	1.24	4.2	38.97
Mean	7.02	6.49	1.01	42.6	4.08	1.26	6.5	41.72
Stand. Dev.	0.19	0.32	0.04	11.3	1.85	0.08	3.5	19.52
Coeff. Var.	0.03	0.05	0.04	0.26	0.45	0.06	0.5	0.47

Table 2: Results of fracture tests for white Carrara marble [18]

Specimen No.	σ_N (MPa)	a_0 (mm)	K_{IC} (MPa√m)	E (GPa)	$a-a_0$ (mm)	K_{IC}^s (MPa√m)	$CTOD_c$ (μm)	Q (mm)
1	7.53	8.74	1.30	53.3	0.92	1.41	2.7	10.49
2	5.08	9.74	0.96	52.9	3.16	1.29	5.1	44.81
3	7.43	8.93	1.30	45.9	0.69	1.38	2.7	7.82
4	3.66	9.74	0.69	42.6	1.25	0.77	2.2	15.09
5	5.07	9.08	0.90	60.3	1.34	1.02	2.1	15.95
6	3.17	10.16	0.62	28.9	0.72	0.66	2.1	8.44
7	7.95	9.59	1.48	50.9	0.68	1.57	2.7	7.76
Mean	5.70	9.43	1.04	47.8	1.25	1.16	2.8	15.77
Stand. Dev.	1.95	0.52	0.33	10.1	0.89	0.35	1.1	13.25
Coeff. Var.	0.34	0.05	0.32	0.21	0.71	0.30	0.4	0.84

Table 3: Results of fracture tests for red Verona marble.

PITFALL OF SIMILITUDE AND SIZE EFFECT

The results of strength and toughness for the two marbles seem to indicate that they have similar mechanical characteristics. This might be used instrumentally in the following to tentatively treat the two marbles as a whole, i.e. as a single material with the same strength and toughness but different material microstructure features and different specimen and crack sizes.

In particular, the nominal peak stress of the notched specimens multiplied by the dimensionless factor in Eq. 2 is plotted against the initial notch length in Fig. 5. Such an adjustment of the nominal stress is for accounting the different relative initial notch lengths in Carrara marble and Verona marble specimens under consideration. In this way, the dependence on $1/\sqrt{a}$ as the crack length is made to vary can be isolated. From the observation of the results in Fig. 5, it can be noted that there is a crack length dependence of the strength, but this deviates from the -0.50 exponent of LEFM. As a matter of fact, considering the mean values of nominal peak stress and the nominal initial notch length for the two marbles, a power law fitting gives us an exponent equal to -0.29 which is in absolute value smaller than that of LEFM. This might be an indication of a deviation from LEFM and, hence, of a pitfall of similitude. In other words, nonlinear phenomena are likely to take place and, therefore, some additional parameters are needed to describe the fracture behaviour of such materials.

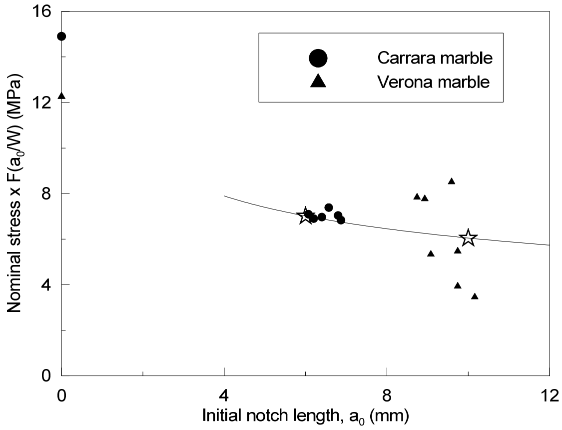

Figure 5: Distribution of the adjusted nominal peak stress against the initial notch length for the two marbles examined. The power law fitting is based on the mean values of strength (star symbols) at nominal initial notch length (fitting slope equal to -0.29). Mean values of tensile strength (see the two symbols at a_0 = 0) are reported as a reference.

MULTI-PARAMETER APPROACHES TO INTERPRET QUASI-BRITTLE BEHAVIOUR

Similar to concrete, which is the archetypal quasi-brittle structural material, natural stones are also characterized by a quasi-brittle fracture behaviour. Therefore, the research findings of the concrete community might be borrowed when natural stones are investigated.

As is well known, the quasi-brittle behaviour is due to, different from the ductile behaviour of metals, the development of a sizable nonlinear zone ahead of the fracture front. Such a zone is almost entirely filled by a Fracture Process Zone (FPZ, see Fig. 3), whose size is equal or proportional to a characteristic (material) length *l*. Depending on the size of the structural component, the FPZ may affect different fractions of the resistant cross-section, or it might be even larger than the cross-section itself. For instance (see Ref. [1]), *l* can encompass order of magnitudes, e.g. from 10-100nm for a silicone waver to about 3m for a dam concrete up to 10m for a grouted soil mass. If D/l (*D* = characteristic size of the resistant cross-section) is, say, greater than 100, the FPZ size is negligible with respect to that of the structure, and LEFM holds. On the contrary, when D/l is in the range of about 5-100, the nonlinear behaviour taking place in the FPZ cannot be disregarded and, hence, size effects intervene and multi-parameter approaches are required when fracture energy and toughness have to be determined.

The simplest conceptual model to characterize the nonlinear behaviour of the finite FPZ is the cohesive model by Hillerborg [21]. Accordingly, the FPZ for Mode I fracture can be described by a fictitious line crack that transmits normal stress σ which is a monotonically decreasing function of the crack opening displacement *w*, namely $\sigma = f(w)$. By definition, $f(0)$ is equal to the tensile strength f_t of the material and, at a certain critical value w_c of the crack opening displacement, the transmitted force across the crack becomes null, i.e. $f(w_c) = 0$. The area underneath the curve $\sigma = f(w)$ is equal to the total energy (G_F) dissipated by fracture per unit area of crack plane, as the crack faces are completely detached at a given point. The quantity G_F has been proved to be equal to the energy dissipated during the crack extension (the latter being the energy which has to be equated, according to Griffith, to the energy release rate of the structure) [22].

Several shapes of the cohesive law $\sigma = f(w)$ are available in the literature, since the decaying exponential law of Hillerborg [21]. The most popular law for concrete is probably a bilinear law presented in Ref. [23]. It is important to stress at this point is that, regardless of the shape of the $\sigma = f(w)$ law, fracture resistance is described by (at least) two parameters (f_t and G_F) due to the sizable FPZ of quasi-brittle materials. In addition, as was pointed out by Planas et al. [24], the initial negative slope in the softening stress-separation law (sometimes it is useful to isolate a fracture energy G_f defined by such an initial slope) controls the peak load of a structure and hence size effect, so that three parameters are needed to identify fracture.

Quasi-brittle behaviour is well captured by cohesive crack models which are nonlinear in nature. This implies that any attempt to directly apply LEFM to quasi-brittle materials needs some adaptation. Therefore, in addition to fracture toughness related to fracture energy through the Irwin relationship ($K_{IC} = \sqrt{E'G_F}$, with $E' = E$ for plane stress and $E' = E/(1-\nu^2)$ for plane strain), at least one material parameter is needed.

The easiest way of adapting LEFM is to define an equivalent crack (regarded as a sharp traction free crack) whose tip is located at some distance ahead the real tip in the FPZ ($a = a_0 + \Delta a$), where Δa is a material parameter. This directly implies that some non-zero crack tip opening displacement (crack blunting) occurs at the end of the length a_0, that is to say, at the beginning of the FPZ.

One possible definition of the material parameter Δa is related to the Irwin-like inelastic zone size $\frac{1}{\pi}\left(\frac{K_{IC}}{f_t}\right)^2$ (e.g. see Ref. [3]). Alternatively, the additional material parameter can be based on the attainment of a critical value of CTOD (w_c) as is suggested by the Jenq-Shah model [2] (that is shown in the next Section).

APPLICATION OF THE TWO-PARAMETER MODEL BY JENQ AND SHAH

This model [2], reminiscent of that by Wells-Cottrell for metals [25], is based on the observation that the initial crack length a_0 in quasi-brittle materials grows slowly well before the peak load is attained. This nonlinear stable stage terminates when the CTOD reaches a critical value (in other words, when the crack propagates to a critical extent) and K_I attains a value K_{IC}^{s} which differs from the nominal K_{IC} (calculated on the basis of a_0). If the geometric and loading conditions are such that, the stress intensity factor is monotonically increasing with the crack length (being the load constant), as the critical condition explained before takes place at the peak load in the case of a 3-point bend beam with an edge crack.
From the value of a, using the LEFM formula, we have:

$$CMOD = w\left(x=0\right) = \frac{4\sigma_N a}{E} V\left(\alpha\right) \tag{3}$$

where

$$V\left(\alpha\right) = 0.76 - 2.28\alpha + 3.87\alpha^2 - 2.04\alpha^3 + \frac{0.66}{\left(1-\alpha\right)^2} \tag{4}$$

The opening displacement along the crack can be expressed as follows [2]:

$$COD = w\left(x\right) = CMOD\sqrt{\left(1-\frac{x}{a}\right)^2 + \left(1.081-1.149\alpha\right)\left[\frac{x}{a}-\left(\frac{x}{a}\right)^2\right]} \tag{5}$$

Considering the compliance parameter C, defined as $C = CMOD/P$, we can write from Eq. 3 (obviously P is equal to $4\sigma_N BW^2/\left(6S\right)$):

$$C = \frac{6Sa}{EBW^2} V\left(\alpha\right) \tag{6}$$

Two compliances measurements, at the initial crack length and at the unknown crack length at peak load, are needed to determine the required parameters. From the former, the Young modulus E is calculated and, from the latter, the effective crack length a at failure is determined. In this way the equivalent crack length (containing the additional parameter $\Delta a = a - a_0$) is estimated and, using the adapted LEFM at the peak load (Eq. 1), the effective toughness K_{IC}^{s} is worked out. This effective toughness is larger than the nominal one based on the initial crack length a_0 and the peak load, and is dependent on the material parameter Δa.
In Fig. 6 for Verona marble, two examples of calculation of initial compliance C_i and failure compliance C_u are shown. In particular, one example is based on the secant compliance at peak load, whereas the other example on the tangent compliance in the unloading-reloading cycle after peak load.
Posing $x = a_0$, we can calculate CTOD from Eq. 6 and, considering the peak load in Eq. 3, the critical value CTOD$_c$ can be obtained.
The additional parameter implicitly introduced in the Jenq-Shah model can be expressed by the length parameter Q, defined as follows [2]:

$$Q = \left(\frac{E \cdot CTOD_c}{K_{IC}^{s}}\right)^2 \tag{7}$$

The results of the two marbles (see Tab. 2 and 3) show that, in Carrara marble, a significant increase of the initial crack length can be observed prior to reaching the critical condition at peak load ($a - a_0$ =4.08mm on average), which is about 20 times the mean calcite grain size (about 200 μm on average), regarded as a characteristic material length. In addition, the length parameter Q is on average equal to 41.72mm, that is, a Q/W = 0.70.
On the other hand, Verona marble exhibits a comparatively smaller crack growth with respect to the initial length, with an extent of 1.25mm on average, which is about 250 times the mean microstructure size of the material (about 5 μm on

average; note that, in this case, the identification of a single material characteristic length from thin section is not straightforward). The length parameter Q is on average equal to 15.77mm, which corresponds to Q/W = 0.53.

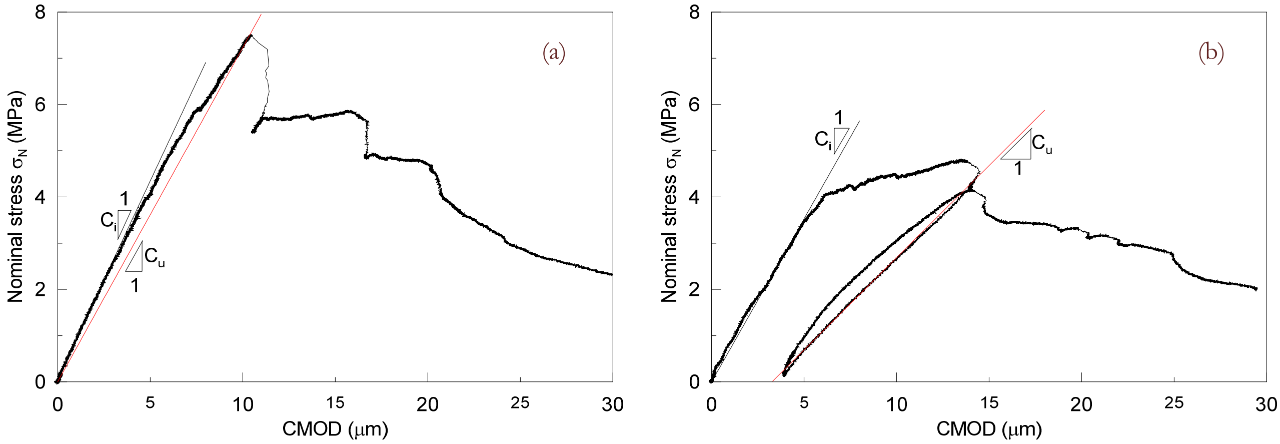

Figure 6: Calculation of the actual crack length for two tests related to Verona marble: (a) compliance at peak load (specimen 1); (b) compliance during unloading-loading cycle in the post-peak phase (specimen 2).

The trend of results shows that there is an increase of both stable crack growth and length parameter Q with the increasing of the characteristic length (of the two marbles) estimated from thin sections technique. This trend is in line with the fact that the size of FPZ is proportional to microstructure characteristic length of the material. In addition, the values of Q/W, together with the relative initial notch length (equal to 0.10 and 0.33 for Carrara marble and Verona marble specimens, respectively), indicate a roughly similar size effect for the two marble specimens [2], as is confirmed by the obtained ratio of nominal against effective critical stress intensity factors (on average equal to 0.80 and 0.89 for Carrara marble and Verona marble specimens, respectively). Instead, the Irwin-like inelastic zone size does not seem to be in line with the trend of microstructure characteristic length of the material (on average equal to 1.45mm and 2.27mm for Carrara marble and Verona marble specimens, respectively).

CONCLUSIONS

Different from the concrete behaviour, the quasi-brittle fracture behaviour of natural rocks has attracted much less attention. In the present paper, the experimental results of the so-called Verona marble are presented and discussed with some previously published results on Carrara marble. Despite their roughly similar nominal tensile strength and fracture toughness, the two marbles are characterized by very different characteristic length of material microstructure, spanning about two order of magnitudes (with Carrara marble having the larger length). A two-parameter model proposed by Shah

and coworkers seems to be suitable to describe the experimental results under discussion. In particular, it is shown that the additional length parameter, represented by the effective crack length, required to describe the nonlinear fracture behaviour, appears to be consistent with the characteristic size of the material microstructure.

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